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Funciones holomorfas de tipo acotado e ideales de polinomios homogéneos en espacios de Banach

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Funciones holomorfas de tipo acotado e ideales de polinomios homogéneos en espacios de Banach

Definimos el concepto de *sucesión coherente* de ideales de polinomios en espacios de Banach, que nos permite relacionar ideales de polinomios homogéneos de diferentes grados. A cada sucesión coherente $\mathfrak{A}$, podemos asociarle un espacio de Fréchet de funciones enteras de tipo acotado, $H_{b\mathfrak{A}}$. Extendemos a $H_{b\mathfrak{A}}$ un resultado de Godefroy y Shapiro sobre hiperciclicidad de operadores de convolución.

También estudiamos el concepto de sucesión multiplicativa de ideales de polinomios a valores escalares. Esto nos permite asociar un álgebra de funciones enteras de tipo acotado $H_{b\mathfrak{A}}$ a cada sucesión coherente y multiplicativa de ideales de polinomios, $\mathfrak{A}$. Probamos que, bajo ciertas condiciones naturales, el espectro del álgebra asociada, $M_{b\mathfrak{A}}$, puede ser dotado de una estructura de dominio de Riemann sobre el bidual del espacio de Banach. Además la extensión de cada función de $H_{b\mathfrak{A}}$ al espectro es una función $\mathfrak{A}$-holomorfa de tipo acotado en cada componente conexa.

Investigamos cómo definir álgebras de funciones holomorfas asociadas a sucesiones de ideales de polinomios en abiertos arbitrarios de un espacio de Banach. Como aplicación probamos que el álgebra de funciones holomorfas nucleares de tipo acotado en un conjunto abierto es un álgebra de Fréchet localmente m-convexa.

Para el álgebra de funciones de tipo acotado, caracterizamos la envoltura holomorfa en término del espectro. Las evaluaciones en puntos de la envoltura son siempre continuas, pero mostramos un ejemplo de un abierto balanceado de c_0 en el que las extensiones a la envoltura no son necesariamente de tipo acotado, respondiendo una pregunta hecha por Hirschowitz. Probamos que para abiertos balanceados y acotados, las extensiones a la envoltura son de tipo acotado.

Palabras clave: Ideales de polinomios, funciones holomorfas de tipo acotado, operadores hipercíclicos, operadores de convolución, envolturas holomorfas, dominios de Riemann.

Holomorphic functions of bounded type and ideals of homogeneous polynomials on Banach spaces

We define the concept of *coherent sequence* of polynomial ideals on Banach spaces, which allows to relate ideals of homogeneous polynomials of different degrees. To each coherent sequence $\mathfrak{A}$, we can associate a Fréchet space of entire mappings of bounded type, $H_{b\mathfrak{A}}$. We extend to $H_{b\mathfrak{A}}$ a result of Godefroy and Shapiro about hypercyclicity of convolution operators.

We also consider the concept of multiplicative sequence of scalar valued polynomial ideals. This allows us to associate an algebra of entire functions of bounded type $H_{b\mathfrak{A}}$ to each coherent and multiplicative sequence of polynomial ideals $\mathfrak{A}$. We prove that, under some natural conditions, the spectrum of the associated algebra, $M_{b\mathfrak{A}}$, can be endowed with a structure of Riemann domain over the bidual of the Banach space. Moreover, the extension of each function in $H_{b\mathfrak{A}}$ to the spectrum is an $\mathfrak{A}$-holomorphic function of bounded type in each connected component.

We investigate how to define algebras of holomorphic functions associated to sequences of polynomial ideals on arbitrary open sets of a Banach space. As an application we show that the algebra of nuclearly holomorphic functions of bounded type on an open set is a locally m-convex Fréchet algebra.

For the algebra of all bounded type functions, we characterize the envelope of holomorphy in terms of the spectrum of the algebra. The evaluations at points of the envelope are always continuous, but we show an example of a balanced open subset of c_0 where the extensions to the envelope are not necessarily of bounded type, answering a question posed by Hirschowitz in 1972. We show that for bounded balanced sets, the extensions to the envelope are of bounded type.

Keywords: Polynomial ideals, holomorphic functions of bounded type, hypercyclic operators, convolution operators, envelopes of holomorphy, Riemann domains.

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Contents

Introduction

In [Gro55], Grothendieck investigated different classes of linear operators, such as nuclear, integral or absolutely summing operators, which played a key role in the development of modern Banach space theory. The investigations of Grothendieck awoke the interest in the possible translation of the classical theory of operators ideals on Hilbert spaces to the Banach space framework. This became a very fruitful area of research, as it can be see, for example, in the monographs [Pie80, DF93]. It should not be a surprise that, in the nonlinear setting, important classes of mappings come out naturally in the same spirit. Indeed, generalizations of the typical classes of linear operator were defined for multilinear mappings and homogeneous polynomials. Let us recall that a function P on a Banach space E is a k-homogeneous polynomial if there exists a k-linear mapping A on $E \times \cdots \times E$ such that $P(x) = A(x, \ldots, x)$ for every $x \in E$. In some sense, the theory of homogeneous polynomials and multilinear mappings may be seen as an extension of the linear theory. In [Pie84], Pietsch took the first step towards a theory of multilinear operator ideals. This notion was immediately adapted to define ideals of homogeneous polynomials, for example, by Braunss in his thesis [Bra84] or Hollstein in [Hol86].

On the other hand, in the sixties the theory of holomorphic functions on infinite dimensional spaces started to develop into a field in its own right. Gupta, in his thesis [Gup68], was interested on differential and convolution operators on spaces of holomorphic functions and needed to define nuclear polynomials and holomorphic functions, which were since then intensively studied. Aware of the importance of having a theory which include as particular cases several classes of holomorphic functions (he was mostly interested in the spaces of continuous holomorphic functions and of nuclear holomorphic functions), Nachbin defined *holomorphy types* (see [Nac69]). A holomorphic function on a Banach space E is, locally, an infinite sum of homogeneous polynomials on E, its Taylor series expansion. Holomorphy types determine spaces of holomorphic functions whose derivatives pertain to a certain class of polynomials $\mathcal{P}_\theta$ (where θ could make reference, for example, to the compact, nuclear or continuous polynomials) and satisfy certain growth conditions relative to the underlying spaces of homogeneous polynomials $\mathcal{P}_\theta$. Shortly after the definition of holomorphy types, several new examples emerged, as the integral [Din71a] or Hilbert-Schmidt [Dwy71] types.

As far as we know, in every example of holomorphy type which appears in the literature, the spaces $\mathcal{P}_\theta$ of polynomials may be thought of as polynomial ideals. Thus, one of the aims of this work is to define a variant of the concept of holomorphy type which is more closely related to the theory of polynomial ideals. After that, we study different classes of holomorphic functions of bounded type on Banach spaces, associated to sequences polynomial ideals. Examples of such spaces of holomorphic functions were already constructed ($\mathcal{L}_\infty$ factorable operators in [Hol86], p-Schatten operators in [Bra92, BJ90]). Also, different authors studied the relationship between holomorphy types and sequences of polynomial ideals with certain properties. For example, it was shown in [Hol86] that a sequence of polynomial ideals constructed via factorization through an ideal of operators is a holomorphy type, and in [BBJP06] the authors proved that if a sequence of ideals has the so called "property B", which is related with stability under differentiation, then it is a holomorphy type and gave some conditions for the converse implication. As we will see, differentiation and multiplication by powers of linear functionals are intrinsic operations for the polynomial ideal structures. Moreover, they are, in some sense, dual to each other, a fact that will be relevant to our study of adjoint polynomial ideals.

We start by relating a linear operator ideal with ideals of homogeneous polynomials of a fixed degree $n > 1$. Many of the well known polynomial ideals can be considered as the n-homogeneous analogous to some operator ideal. This is the case, for example, of the ideals of nuclear, integral or compact polynomials. However, the extension of a linear operator ideal to higher degrees is not always obvious. For example, many extensions of the ideal of absolutely r-summing operators have been developed, among them, the absolutely, the multiple and the strongly r-summing polynomials and the r-dominated polynomials.

In order to shed some light on what makes a particular n-homogeneous extension of an operator ideal natural, we introduce the concept of *compatibility* between a polynomial ideal and an operator ideal. An n-homogeneous polynomial ideal $\mathfrak{A}_n$ and an operator ideal $\mathfrak{A}$ turn out to be compatible if any time we take a polynomial $P \in \mathfrak{A}_n$ and we fix $n-1$ variables of its associated n-linear map, the resulting operator is in $\mathfrak{A}$ and if every operator in $\mathfrak{A}$ multiplied by $n-1$ linear functionals is a polynomial in $\mathfrak{A}_n$. Note that the operation of fixing variables to the n-linear map is analogous to differentiating P. Compatibility relates each polynomial ideal with one operator ideal; that is, it is proved that a Banach ideal of n-homogeneous polynomials is compatible with one and only one Banach ideal of operators. Most (but not all) ideals of polynomials usually studied are shown to be compatible with the ideal of operators one would expect. On the other hand, given an ideal of operators $\mathfrak{A}$, there are many polynomial ideals compatible with it. Indeed, there is a greatest and a smallest polynomial ideal compatible with $\mathfrak{A}$, which are always different as polynomial ideals. It is also shown that compatibility is preserved under several natural ideal operations, such as taking adjoints, maximal or minimal hulls and composing with some operator ideal.

The polynomial extension of an operator ideal $\mathfrak{A}$ usually gives rise to a sequence of polynomial ideals $\mathfrak{A} = \{\mathfrak{A}_k\}_k$, each $\mathfrak{A}_k$ an ideal of k-homogeneous polynomials. Therefore, it is also interesting to study the relationship between different $\mathfrak{A}_k$'s. To this end, the concept of *coherence* of a sequence of polynomial ideals is also introduced. A coherent sequence $\mathfrak{A}$ allow us to define a space of entire mappings of bounded type $H_{b\mathfrak{A}}$ associated to $\mathfrak{A}$. As particular cases of the spaces $H_{b\mathfrak{A}}$ we have the classical space of all continuous holomorphic functions of bounded type H_b, and the spaces of holomorphic functions of nuclear [Gup70], weakly continuous on bounded sets [Aro79], integral [DGMZ04] or Hilbert-Schmidt [Dwy71, Pet01] bounded type, among others. In these spaces we will address questions about the Borel transforms and duality. Whenever the ideals of the sequence are minimal we will describe the dual of $H_{b\mathfrak{A}}$ as a space of holomorphic functions of exponential type. This will allow us to characterize convolution operators and to prove that they are hypercyclic whenever they are not a scalar multiple of identity, extending a theorem of Godefroy and Shapiro [GS91]. Most of the spaces of holomorphic functions mentioned are in fact algebras, so it is also interesting to investigate when the spaces $H_{b\mathfrak{A}}$ are algebras. With this goal in mind, we define *multiplicative sequences* as an extension of the concept of coherence. If a coherent sequence is also multiplicative, $H_{b\mathfrak{A}}$ becomes an algebra, and, in those cases we will study its spectrum. In many cases, it is shown that the spectrum has a structure of analytic manifold (modeled on the bidual of the base space), and that the functions in $H_{b\mathfrak{A}}$ extend analytically to it.

Finally, we concentrate in the algebra of all analytic functions of bounded type on a general domain U. We focus on the problem of finding the largest open set to which all those functions uniquely extend and to determine whether these extensions are of bounded type. As it could be expected, to properly pose and study the problem, we must expand our investigations to the Riemann domains framework. Loosely speaking, if X is a Riemann domain over the Banach space E, the H_b-envelope of holomorphy of X is the largest Riemann domain (over E) "containing X" to which every holomorphic function of bounded type on X has a unique extension. Our problem translates, then, to the characterization of the H_b-envelope of holomorphy of a Riemann domain modeled on a Banach space E. In several complex variables it is well known that the envelope of holomorphy of a domain X is the spectrum of the algebra of holomorphic functions on X, $H(X)$. The envelope of holomorphy for the space of all holomorphic functions on a Riemann domain over a Banach space was first constructed by Hirschowitz [Hir72] using germs of holomorphic functions. There he also showed that this construction could be adapted to obtain the H_b-envelope, that is, the envelope of holomorphy for the space of bounded type functions. He also asked wether the extensions to the H_b-envelope should necessarily be of bounded type or not. We will answer this question by the negative. To obtain this answer we need to characterize the H_b-envelope of holomorphy of X in terms of the spectrum of $H_b(X)$. Under the hypothesis of symmetric regularity, the spectrum was shown in [AGGM96, DV04] to be a Riemann domain over the bidual of the base space E. Thus, in general, the spectrum cannot be the H_b-envelope of holomorphy of X. Even in the case that E is reflexive, the spectrum is usually too large to be the H_b-envelope of X. For example, in the case $X = E$, the spectrum may have an infinite number of connected components. Still, it is proved that the H_b-envelope may be identified with a part of the spectrum, and this is achieved without the assumption of symmetric regularity of E. We can then obtain a simpler characterization of the H_b-envelope for a balanced open subset U, in terms of its polynomial hull and to prove that if U is also bounded then the extensions of bounded type functions on U to the H_b-envelope of U are of bounded type. However, we will show an example of an unbounded balanced open set U and a bounded type function on U, such that its extension to the H_b-envelope of U is not of bounded type.

We now describe the contents of each chapter of this thesis.

Chapter 1: Preliminaries

In the first chapter, we define the basic concepts and describe some properties about polynomials, polynomial ideals, tensor norms and holomorphic mappings on Banach spaces which we will need in the rest of this work.

Chapter 2: Compatible ideals

In this chapter we define the compatibility of a quasi-normed ideal of homogeneous polynomials and a quasi-normed ideal of linear operators. Most examples of polynomials ideals where defined as generalizations of an ideal of linear operators and, in almost every case, they are compatible with that ideal of operators. For instance, the ideals of n-homogeneous nuclear, integral, extendible or approximable polynomials are compatible with the correspondent ideal of operators. However, the ideal of absolutely p-summing polynomials is not compatible with the ideal of absolutely p-summing operators. This fact has the following consequence: if $n \geq 2$, then every absolutely summing n-homogeneous polynomial from E to E is weakly compact if and only if E is reflexive

(Corollary 2.1.25). In contrast, in the linear case, it is well known that every absolutely summing linear operator on a Banach space E is weakly compact.

It is shown that there are many polynomial ideals compatible with a given operator ideal $\mathfrak{A}$. Moreover an n-homogeneous polynomial ideal $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$ if and only if $\mathcal{F}_n^{\mathfrak{A}} \hookrightarrow \mathfrak{A}_n \hookrightarrow \mathcal{M}_n^{\mathfrak{A}}$, where $\mathcal{F}_n^{\mathfrak{A}}$ and $\mathcal{M}_n^{\mathfrak{A}}$ are, respectively, the smallest and the greatest ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$ (Section 2.2). In the following sections the compatibility is shown to be preserved by several procedures usually performed with ideals. It is proved that:

- If we compose compatible ideals with closed operator ideals, we obtain compatible ideals (Proposition 2.3.1).
- The interpolation of compatible ideals gives compatible ideals (Proposition 2.4.3).
- If $\mathfrak{A}$ and $\mathfrak{A}_n$ are compatible, so are their adjoints ideals (Proposition 2.6.1) and their maximal and minimal hulls (Corollaries 2.6.3 and 2.6.2).

The theory of polynomial and operators ideals is closely related to the theory of tensor products of Banach spaces. So we also investigate which are the conditions which relate the tensor norms associated to compatible ideals. The concept of compatibility helps us to prove a conjecture of Floret and Hunfeld [FH02] about the existence of certain mixed tensor norms. This is contained in Section 2.5.

In the last section we show that compatibility may be applied to obtain some polynomial characterizations of Banach spaces. For example, the compatibility may be used to prove that a Banach space E is Asplund if and only if every Pietsch integral polynomial on E is nuclear (this result was originally proven in [CG04], see Corollary 2.7.3).

Chapter 3: Coherent sequences and holomorphic mappings

We define the concept of coherent sequence of polynomial ideals. This is an adaptation of the concept of compatibility in order to relate polynomial ideals of different degrees. Indeed, $\{\mathfrak{A}_k\}_{k=1}^{\infty}$ is a coherent sequence if for every $P \in \mathfrak{A}_k(E,F)$, $a \in E$ and $\gamma \in E'$, the polynomial $d^{k-1}P(a)$ is in $\mathfrak{A}_{k-1}(E,F)$ and the polynomial γP belongs to $\mathfrak{A}_{k+1}(E,F)$. In addition, there is a condition that control the norms of $d^{k-1}P(a)$ and γP. Moreover, since the only scalar ideal of linear operators is E', compatibility is a trivial concept in the scalar case. But there are many interesting examples of coherent sequences of scalar valued ideals.

The chapter is divided in two sections. In the first one, we give examples of coherent sequences, we show that coherence is preserved by several operations of the ideals and we relate the coherence of a sequence with properties of the associated tensor norms. Many of these properties and examples are similar to the ones given in the previous chapter, and are thus shown only once.

In the second section, given a coherent sequence $\mathfrak{A} = \{\mathfrak{A}_k\}$, we define a Fréchet space of holomorphic functions of bounded type $H_{b\mathfrak{A}}$ associated to it. A holomorphic function f is in $H_{b\mathfrak{A}}(E,F)$ if the polynomials of the Taylor series of f at 0 belong to the $\mathfrak{A}_k$'s and the series has an infinite "$\mathfrak{A}$-radius of convergence". Several spaces of holomorphic functions that were previously studied are particular cases of the spaces of functions we define. For example, bounded type holomorphic functions of nuclear, Hilbert-Schmidt or integral type.

Given $f \in H_b(E)$ and $\varphi \in H_b(E)'$ the product $\varphi * f \in H_b(E)$ is defined in [ACG91] by $\varphi * f(x) = \varphi(f(x+\cdot))$. For the spaces $H_{b\mathfrak{A}}$ it is shown that:

- Under the additional assumption of *weakly differentiability* (Definition 3.2.15) of the coherent sequence $\mathfrak{A}$, the application $T_\varphi(f) = \varphi * f$ is continuous in $H_{b\mathfrak{A}}(E)$ (Theorem 3.2.17).

- Any convolution operator in $H_{b\mathfrak{A}}(E)$, that is, an operator which commutes with translations, is of the form T_φ for some $\varphi \in H_{b\mathfrak{A}}(E)'$ (Corollary 3.2.18).

We also prove that most of the examples of coherent sequences are weakly differentiable.

The Borel transform for a space $\mathfrak{A}_k(E)$ of k-homogeneous polynomials is the mapping $B_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$, defined by $B_k(\varphi)(\gamma) = \varphi(\gamma^k)$. In many cases, the dual of a space of polynomials may be identified, through the Borel transform, with another space of polynomials. This is the case of nuclear, approximable, Hilbert-Schmidt polynomials and, in general, any minimal ideal of polynomials. In these cases, we characterize the dual of $H_{b\mathfrak{A}}(E)$ as a space of exponential type holomorphic functions on E'. This is applied to obtain the following generalization to infinite dimensional spaces of a theorem of Godefroy and Shapiro [GS91] about the hypercyclicity of convolution operators on $H(\mathbb{C}^n)$:

- If E' is separable and $\{\mathfrak{B}_k(E')\}_k$ is a coherent sequence and $\{\mathfrak{A}_k(E)\}_k$ is such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k, then every convolution operator $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ which is not a scalar multiple of the identity is hypercyclic (Theorem 3.2.39).

Some results of [AB99, Pet01, Pet06] are particular cases of the above theorem.

We also introduce the space of Schatten-von Neumann bounded type functions using interpolation theory (based on prior work in [CKP92] on multilinear forms), and show that the above result may be applied to that space.

At the end of this chapter, we show how bounded type functions associated to a coherent sequence may be defined on balls and more general domains of Banach spaces.

Chapter 4: Multiplicative sequences and algebras of holomorphic functions

In this chapter we are interested in algebras of holomorphic functions of bounded type associated to sequences of polynomial ideals. It is an immediate consequence of the definition that if $\mathfrak{A}$ is a coherent sequence then the product of a polynomial in $\mathfrak{A}$ and some power of a linear functional is again in $\mathfrak{A}$. However, the product of two polynomials in $\mathfrak{A}$ is not necessarily in $\mathfrak{A}$ (see Example 4.1.1). We thus introduce *multiplicative sequences* of polynomial ideals, which are both coherent and closed under products of polynomials. When $\mathfrak{A}$ is a multiplicative sequence, then $H_{b\mathfrak{A}}(E)$ becomes a B_0-algebra. In the final section of this chapter we will show that several examples are actually locally multiplicatively convex Fréchet algebras. Almost every example of coherent sequence considered so far is also multiplicative. Moreover, multiplicativity is preserved by interpolation of ideals, taking maximal and minimal hulls and composition with closed ideals of operators. Multiplicativity is not preserved by taking adjoints. To obtain the multiplicativity of the adjoint sequence, we should have a property "dual" to being closed under multiplication of polynomials. At first sight, it is not clear what this property should be. Surprisingly, weakly differentiability, which was defined in the previous chapter to deal with convolution operators, is the desired property. Indeed, if a sequence of polynomial ideals is weakly differentiable, then the sequence of adjoints ideals is multiplicative (Proposition 4.1.17). The converse is true if we have density of finite type polynomials. We also relate the multiplicativity condition with properties of the associated tensor norms, and we show that polynomial ideals associated to natural symmetric tensor norms (in the sense of [CGd]) are multiplicative.

In the second section we use multiplicativity to obtain another characterization of convolution operators on $H_{b\mathfrak{A}}$.

In the third section we study the spectrum $M_{b\mathfrak{A}}(E)$ of the algebra $H_{b\mathfrak{A}}(E)$. It is shown in [AGGM96] that whenever E is symmetrically regular the spectrum of $H_b(E)$ (the algebra of all holomorphic functions of bounded type) is a Riemann domain spread over the bidual E'' (actually they show this fact for functions defined on arbitrary open sets of E). Moreover, in [Din99, Section 6.3] it is proved that the extensions to each connected component of the spectrum may be considered a function of bounded type. We establish those results for $\mathfrak{A}$-entire functions of bounded type, for several multiplicative sequences $\mathfrak{A}$. As in the case of H_b, the Aron-Berner extension plays a crucial role there. So we begin by studying when a sequence is closed under the Aron-Berner extension (or AB-closed). Symmetric regularity was used in [AGGM96] to obtain symmetric Aron-Berner extensions of multilinear forms. But most sequences of ideals of polynomials (different to $\{\mathcal{P}^k\}$) are *regular*, that is, the multilinear forms associated to polynomials have symmetric Aron-Berner extensions. Thus, the assumption of symmetric regularity on the space is not necessary for most of our results:

- Let $\mathfrak{A}$ be an AB-closed multiplicative sequence which is regular at a Banach space E. Then $(M_{b\mathfrak{A}}(E), \pi)$ is a Riemann domain over E'' and each connected component of $(M_{b\mathfrak{A}}(E), \pi)$ is homeomorphic to E'' (Theorem 4.3.14).

- If $\mathfrak{A}$ is also weakly differentiable at E, then, for every function $f \in H_{b\mathfrak{A}}(E)$, the extension $\tilde{f}$ to $M_{b\mathfrak{A}}(E)$ results an $\mathfrak{A}$-holomorphic function of bounded type when restricted to each connected component of $M_{b\mathfrak{A}}(E)$ (Theorem 4.3.19).

These results may be applied for example when $\mathfrak{A}$ is the sequence of integral or extendible polynomials, or the maximal ideals associated to natural tensor norms. Finally, we address a Banach-Stone type question on these algebras: if $H_{b\mathfrak{A}}(E)$ and $H_{b\mathfrak{B}}(F)$ are (topologically and algebraically) isomorphic, what can we say about E and F? We obtain results in this direction which allow us to show, for example, that if E or F is reflexive and $\mathfrak{A}$ and $\mathfrak{B}$ are any of the sequences of nuclear, integral, approximable or extendible polynomials, then if $H_{b\mathfrak{A}}(E)$ is isomorphic to $H_{b\mathfrak{B}}(F)$ it follows that E and F are isomorphic.

In the last section, we investigate conditions of the sequence of ideals $\mathfrak{A}$ under which the spaces $H_{b\mathfrak{A}}(U)$ of $\mathfrak{A}$-holomorphic functions of bounded type on an open set are an algebra. To achieve this, we need to seek for better bounds on the norms of products of homogeneous polynomials in the ideals $\mathfrak{A}_k$. The obtained bounds will allow us to show that in several cases the spaces $H_{b\mathfrak{A}}$ are locally m-convex algebras.

Chapter 5: Envelopes of holomorphy

In this last chapter the algebra H_b of bounded type holomorphic functions on general domains is studied in more detail. We consider the problem of extending holomorphic functions of bounded type defined on an open subset U of a Banach space, to larger domains and determining if these extensions are also of bounded type.

In the first section we study the H_b-envelope of holomorphy of a Riemann domain. We also consider two alternative definitions of the envelope: the first one requires that extensions be also of bounded type (we call it the H_b-H_b-envelope). The second one requires that evaluations on points of the envelope be continuous functionals on $H_b(X)$ (we call this one the strong H_b-envelope).

Although the spectrum is known to have an analytic structure only in the symmetrically regular case, we are able to give a characterization of the strong H_b-envelope of holomorphy of X as a subset of the spectrum $M_b(X)$, much in the spirit of the several complex variables theory, for domains over arbitrary Banach spaces. We show that the H_b-envelope and the strong H_b-envelope coincide (Theorem 5.1.7), and that whenever the H_b-H_b-envelope exists, it must also coincide with the classical H_b-envelope (Theorem 5.1.11).

In the second section we study extensions of holomorphic functions of bounded type on an open subset of E. We give a precise description of the H_b-envelope of a balanced open set U, which turns out to be a (possibly larger) open subset of E. We prove some good properties of the extensions of functions of $H_b(U)$ to the envelope. In particular, we show:

- Let U be a bounded balanced open set and $f \in H_b(U)$. Then the extension of f to the H_b-envelope of U is of bounded type (Theorem 5.2.11).

However, this is not true for unbounded balanced open sets:

- There exist an unbounded open balanced subset U of c_0, and a function $f \in H_b(U)$ such that the extension of f to the H_b-envelope of U is not of bounded type (Example 5.2.8).

This example answers a question posed by Hirschowitz [Hir72, Remarque 1.8]. This, in particular, also shows that the H_b-H_b-envelope does not always exist and that the canonical extension of a function of bounded type to the spectrum of $H_b(U)$ is not necessarily of bounded type.

The entire functions of bounded type on a Banach space extend naturally to the bidual via the Aron-Berner extension [AB78]. Thus it is also a natural problem to find the largest set of E'' to which every bounded type function on a given set $U \subset E$ extends. In section 3 we address this problem. We define the AB-H_b-envelope of a domain U on E, which is, roughly speaking, the largest domain Y over E'' such that every bounded type function on U extend uniquely to Y in such a way that this extension coincides locally with the Aron-Berner extension. We give a characterization of this envelope in the case U is an open and balanced subset of a symmetrically regular space.

In section 4 we consider Banach spaces for which finite type polynomials are dense in $H_b(E)$. When they are also reflexive, they are called Tsirelson-like spaces following [Vie07]. We characterize the density of finite type polynomials in terms of the spectrum of $H_b(U)$ (more precisely, in terms of $\pi(M_b(U))$, the projection of the spectrum on E''). We also show that Tsirelson-like spaces are precisely the spaces where the holomorphic convexity of some U is equivalent to all the elements of the spectrum being evaluations on points in U, extending some results of [Muj01] and [Vie07]. This means that Tsirelson-like spaces are the only spaces that behave as in the several complex variables theory. We also give a Banach-Stone type result which generalizes some results in [Vie07] and [CGM05].

In the last section we present some properties of the spectrum of $H_b(U)$, somehow extending the study of [AGGM96] and [CGM05]. Even though, for a symmetrically regular space E, the extension of a bounded type entire function to the spectrum $M_b(E)$ is of bounded type on each sheet of the spectrum, we prove that usually it is not of bounded type on the whole spectrum. More precisely, our result is:

- On any symmetrically regular Banach space in which there is a polynomial which is not weakly continuous on bounded sets, there exist homogeneous polynomials whose extensions are not of bounded type on the whole spectrum $M_b(E)$ (Proposition 5.5.2).

Then we concentrate in the case $U = B_{\ell_p}$ to show that the structure of the spectrum in not what one may expect from the case $U = E$, with E a symmetrically regular Banach space. In the latter case, $M_b(E)$ is the disjoint union of copies of E''. However, we show that $M_b(B_{\ell_p})$ is not a disjoint union of "unit balls". Finally, for $p \in \mathbb{N}$, we are able to distinguish a part of the spectrum where the canonical extensions are of bounded type and which turns out to be a H_b-domain of holomorphy.

Chapter 1

Preliminaries

1.1 Polynomials on Banach spaces

Throughout this work E, F and G will be complex Banach spaces. B_E and S_E will denote the unit ball and the unit sphere of E, respectively.

Definition 1.1.1. Let $n \in \mathbb{N}$. An application $P : E \to F$ is an *n-homogeneous polynomial* if there exist an n-linear mapping $\Phi : E \times \overset{k}{\ldots} \times E \to F$ such that $P(x) = \Phi(x, \ldots, x)$ for every $x \in E$. In this case we will say that P is a polynomial associated to Φ and denote $P = \hat{\Phi}$.

Given a polynomial P there are many n-linear mappings which satisfy condition on Definition 1.1.1, but there exists only one which is symmetric (an n-linear mapping Φ is symmetric if $\Phi(x_1, \ldots, x_n) = \Phi(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every $x_1, \ldots, x_n$ and every permutation σ of $\{1, \ldots, n\}$). This symmetric n-linear form, which will be denoted by $\overset{\vee}{P}$, may be obtained from P via the **polarization formula**:

$$\overset{\vee}{P}(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n P\big(\sum_{i=1}^{n} \varepsilon_i x_i\big).$$

Conversely, to each symmetric n-linear form we can associate an n-homogeneous polynomial. Thus there exist a one to one and onto correspondence between n-homogeneous polynomials and n-linear symmetric forms. We also denote $T_P : \bigotimes^{n,s} E \to F$ the linearization of P:

$$T_P\Big(\sum_j x_j \otimes \cdots \otimes x_j\Big) = \sum_j P(x_j).$$

The following norm is defined for n-homogeneous polynomials:

$$\|P\| = \sup_{x \in B_E} \|P(x)\|.$$

An n-homogeneous polynomial P is continuous if and only if $\|P\| < \infty$. It is easy to see that $\|P\|$ is the least constant such that $\|P(x)\| \leq \|P\|\|x\|^n$ for every $x \in E$. We will denote by $\mathcal{P}^n(E, F)$ the Banach space of all continuous n-homogeneous polynomials from E to F, or $\mathcal{P}^n(E)$ when $F = \mathbb{C}$. Then $(\mathcal{P}^n(E, F), \| \cdot \|)$ is a Banach space. We will convey that $\mathcal{P}^0(E, F) = F$.

Denote by $\mathcal{L}_s^n(E, F)$ the space of continuous n-linear symmetric forms from E to F. It is a Banach space with the norm $\|\Phi\| = \sup\{\|\Phi(x_1, \ldots, x_n)\| : x_1, \ldots, x_n \in B_E\}$. Then the polarization formula implies that

$$\|P\| \leq \|\overset{\vee}{P}\| \leq \frac{n^n}{n!}\|P\|.$$

Thus the application

$$\begin{array}{ccc} \mathcal{P}^n(E,F) & \to & \mathcal{L}^n_s(E,F) \\ P & \mapsto & \overset{\vee}{P} \end{array}$$

is a Banach space isomorphism.

Example 1.1.2. Maybe the simplest class of polynomials is the class of *finite type* polynomials, $\mathcal{P}^n_f(E,F)$. An n-homogeneous polynomial P is of finite type if there exist $\gamma_1,\dots,\gamma_k \in E'$, $y_1,\dots,y_k \in F$ such that $P(x) = \sum \gamma_j(x)^n y_j$ for every x in E. If E is finite dimensional then every polynomial on E is of finite type. The closure of finite type polynomials in $\mathcal{P}^n(E,F)$ are the *approximable* polynomials. The space of approximable polynomials is denoted by $\mathcal{P}^n_A(E,F)$. Every homogeneous polynomial on c_0 is approximable (see [Din99, Propositions 1.59 and 2.8]), but there, in general, are plenty of non approximable polynomials. For example, the 2-homogeneous polynomial $P(x) = \sum_k x_k^2$ on ℓ_2 is not approximable.

Given $P \in \mathcal{P}^n(E,F)$ and $a \in E$, we define the polynomial $P_{a^k} \in \mathcal{P}^{n-k}(E,F)$ by

$$P_{a^k}(x) = \overset{\vee}{P}(a^k, x^{n-k}) = \overset{\vee}{P}(\overbrace{a,\dots,a}^{k}, \overbrace{x,\dots,x}^{n-k}).$$

We say that P_{a^k} is the polynomial obtained from P by fixing k variables at a. For $k=1$, we write P_a instead of P_{a^1}. The k-differential of a polynomial is the application $d^kP : E \to \mathcal{P}^k(E,F)$ defined by

$$\frac{d^kP(x)}{k!}(y) = \binom{n}{k} \overset{\vee}{P}(x^{n-j}, y^j).$$

Then we have that $P(x+y) = \sum_{k=0}^n \binom{n}{k} \overset{\vee}{P}(x^{n-k}, y^k) = \sum_{k=0}^n \frac{d^kP(x)}{k!}(y)$.

1.2 Ideals of homogeneous polynomials

The definition of polynomial ideals appeared first in [Bra84, Hol86] as an adaption of the definition of ideals of multilinear mappings given by Pietsch [Pie84] (for more on this subject see [Flo01, Flo02, FGa03, FH02]). A **quasi-normed ideal of continuous n-homogeneous polynomials** is a pair $(\mathfrak{A}_n, \|\cdot\|_{\mathfrak{A}_n})$ such that for each Banach spaces E, F, E_1, F_1:

(i) $\mathfrak{A}_n(E,F) = \mathfrak{A}_n \cap \mathcal{P}^n(E,F)$ is a linear subspace of $\mathcal{P}^n(E,F)$ and $\|\cdot\|_{\mathfrak{A}_n(E,F)}$ is a quasi-norm on it.

(ii) If $T \in \mathcal{L}(E_1,E)$, $P \in \mathfrak{A}_n(E,F)$ and $S \in \mathcal{L}(F,F_1)$, then $S \circ P \circ T \in \mathfrak{A}_n(E_1,F_1)$ and

$$\|S \circ P \circ T\|_{\mathfrak{A}_n(E_1,F_1)} \le \|S\| \|P\|_{\mathfrak{A}_n(E,F)} \|T\|^n$$

(iii) $z \mapsto z^n$ belongs to $\mathfrak{A}_n(\mathbb{C},\mathbb{C})$ and has norm 1.

We will also frequently use the notion of **normed scalar ideal of continuous n-homogeneous polynomials**, which is a pair $(\mathfrak{A}_n, \|\cdot\|_{\mathfrak{A}_n})$ such that for each Banach spaces E, E_1:

(i) $\mathfrak{A}_n(E) = \mathfrak{A}_n \cap \mathcal{P}^n(E)$ is a linear subspace of $\mathcal{P}^n(E)$ and $\|\cdot\|_{\mathfrak{A}_n(E)}$ is a norm on it.

(ii) If $T \in \mathcal{L}(E_1,E)$ and $P \in \mathfrak{A}_n(E)$, then $P \circ T \in \mathfrak{A}_n(E_1)$ and

$$\|S \circ P \circ T\|_{\mathfrak{A}_n(E_1)} \le \|P\|_{\mathfrak{A}_n(E)} \|T\|^n$$

(iii) $z \mapsto z^n$ belongs to $\mathfrak{A}_n(\mathbb{C})$ and has norm 1.

It is well-known that the only scalar ideal of 1-homogeneous polynomials (that is, of linear functionals) is, for each Banach space E equal to E'.

We now recall the definition of the ideals of polynomials which may be encountered in the following chapters.

- *Continuous polynomials*, $\mathcal{P}$.

  The ideal of all continuous polynomials, with the usual norm of polynomials is a Banach ideal of homogeneous polynomials. Other polynomial ideals with the usual norm of polynomials are:

  - *Finite type polynomials*, $\mathcal{P}_f$ and *approximable polynomials*, $\mathcal{P}_A$, which were already defined in Example 1.1.2.
  - *Compact polynomials*, $\mathcal{P}_K$ and *weakly compact polynomials*, $\mathcal{P}_{WK}$.
    A polynomial $P \in \mathcal{P}^n(E, F)$ is (weakly) compact if it maps bounded sets of E on relatively (weakly) compact sets on F. $\mathcal{P}_K$ and $\mathcal{P}_{WK}$ are complete normed polynomial ideals.
  - *Weakly continuous on bounded sets polynomials*, $\mathcal{P}_w$.
    A polynomial $P \in \mathcal{P}^n(E, F)$ is weakly continuous on bounded sets if the restriction of P to any bounded set of E is continuous when the weak topology is considered on E and the norm topology on F.
  - *Weakly sequentially continuous polynomials*, $\mathcal{P}_{wsc}$.

- *Nuclear polynomials*, $\mathcal{P}_N$.

  A polynomial $P \in \mathcal{P}^k(E; F)$ is said to be **nuclear** if it can be written as $P(x) = \sum_{i=1}^{\infty} \gamma_i(x)^k y_i$, where $\gamma_i \in E'$, $y_i \in F$ for all i and $\sum_{i=1}^{\infty} \|\gamma_i\|^k \|y_i\| < \infty$. The space of nuclear k-homogeneous polynomials from E into F will be denoted by $\mathcal{P}^k_N(E; F)$. It is a Banach space when we consider the norm

  $$\|P\|_{\mathcal{P}^k_N(E;F)} = \inf \left\{ \sum_{i=1}^{\infty} \|\gamma_i\|^k \|y_i\| \right\}$$

  where the infimum is taken over all representations of P as above.

- *Integral polynomials*, $\mathcal{P}_{PI}$ and $\mathcal{P}_{GI}$.

  A polynomial $P \in \mathcal{P}^k(E, F)$ is **Pietsch-integral** if there exists a regular F-valued Borel measure μ, of bounded variation on $(B_{E'}, w^*)$ such that

  $$P(x) = \int_{B_{E'}} \gamma(x)^n \, d\mu(\gamma)$$

  for all $x \in E$. The space of k-homogeneous Pietsch-integral polynomials is denoted by $\mathcal{P}^k_{PI}(E, F)$ and the integral norm of a polynomial $P \in \mathcal{P}^k_{PI}(E, F)$ is defined as

  $$\|P\|_{\mathcal{P}^k_{PI}(E,F)} = \inf \{|\mu|(B_{E'})\},$$

  where the infimum is taken over all measures μ representing P.

  The definition of **Grothendieck-integral** polynomials is analogous, but taking the measure μ to be F''-valued. The space of Grothendieck-integral polynomials is denoted by $\mathcal{P}^k_{GI}(E, F)$. For scalar valued polynomials, $\mathcal{P}_{GI} = \mathcal{P}_{PI}$ and will be denoted by $\mathcal{P}_I$.

- *Extendible polynomials*, $\mathcal{P}_e$.

  A polynomial $P : E \to F$ is **extendible** if for any Banach space G containing E there exists $\widetilde{P} \in \mathcal{P}^k(G, F)$ an extension of P. We will denote the space of all such polynomials by $\mathcal{P}_e^k(E, F)$. For $P \in \mathcal{P}_e^k(E, F)$, its extendible norm is given by

$$\|P\|_{\mathcal{P}_e^k(E,F)} = \inf\{c > 0 : \quad \text{for all } G \supset E \text{ there is an extension of } P \text{ to } G \text{ with norm} \leq c\}.$$

- *Multiple r-summing polynomials*, $\mathcal{M}_r$.

  We need to recall the definition of the weak r-norm of a sequence: for $x^1, \ldots, x^m \in E$, we define

$$w_r\big((x^i)_{i=1}^m\big) = \sup_{\gamma \in B_{E'}} \left(\sum_i |\gamma(x^i)|^r \right)^{1/r}.$$

  A k-homogeneous polynomial P from E to F is **multiple r-summing** if there exists $C > 0$ such that for every choice of finite sequences $(x_j^{i_j})_{i_j=1}^{m_j} \subset E$, $j = 1, \ldots, k$, the following holds

$$\left(\sum_{i_1,\ldots,i_k=1}^{m_1,\ldots,m_k} \|\overset{\vee}{P}(x_1^{i_1}, \ldots, x_k^{i_k})\|^r \right)^{\frac{1}{r}} \leq C \cdot w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_k^{i_k})_{i_k=1}^{m_k}).$$

  The least of such constants C is called the **multiple r-summing norm** and denoted $\|P\|_{\mathcal{M}_r^k(E,F)}$.

- *r-dominated polynomials*, $\mathcal{D}_r$.

  A k-homogeneous polynomial P from E to F is **r-dominated** if there exists $C > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \|P(x^i)\|^{\frac{r}{k}} \right)^{\frac{k}{r}} \leq C \cdot w_r((x^i)_{i=1}^m)^k.$$

  The least of such constants C is called the **r-dominated (quasi) norm** (which is a norm for $r \geq n$) and denoted $\|P\|_{\mathcal{D}_r^k(E,F)}$.

- *Absolutely p-summing polynomials*, Π_p.

  A k-homogeneous polynomial P from E to F is **absolutely p-summing** if there exists $C > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \|P(x^i)\|^p \right)^{\frac{1}{p}} \leq C \cdot w_p((x^i)_{i=1}^m)^k.$$

  The least of such constants C is called the **absolutely p-summing norm**.

- *Strongly p-summing polynomials*, $\mathcal{S}_p$.

  A k-homogeneous polynomial P from E to F is **strongly p-summing** if there exists $C > 0$ such that for every finite sequence $(x^i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \|P(x^i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{Q \in B_{\mathcal{P}^k(E)}} \left(\sum_i |Q(x^i)|^p \right)^{1/p}.$$

  The least of such constants C is called the **strongly p-summing norm**.

- *r-factorable polynomials*, $\mathcal{L}_r$ and *strongly r-factorable polynomials*, $\mathcal{SL}_r$.

  A k-homogeneous polynomial P from E to F is r-**factorable** (strongly r-factorable) if there exist a measure space (Ω, μ), a linear operator $S \in \mathcal{L}(E, L_r(\Omega))$ and $Q \in \mathcal{P}^k(L_r(\Omega), F'')$ $(Q \in \mathcal{P}^k(L_r(\Omega), F))$ such that $P = Q \circ S$ $(J_F \circ P = Q \circ S)$. The quasi-norm considered is the infimum of $\|Q\|_{\mathcal{P}^k} \|S\|_{\mathcal{L}}^k$ over all factorizations of P.

- *∞-compact polynomials*: $\mathcal{K}_\infty$.

  A k-homogeneous polynomial P from E to F is ∞-**compact** if there exist a linear operator $S \in \mathcal{L}(E, c_0)$ and $Q \in \mathcal{P}^k(c_0, F)$ such that $P = Q \circ S$. The norm considered is the infimum of $\|Q\|_{\mathcal{P}^k} \|S\|_{\mathcal{L}}^k$ over all factorizations of P.

There are also several ways to construct new polynomial ideals from given ones. We describe some of the procedures which will be used later.

Composition Ideals

Let $\mathfrak{A}_n$ be an ideal of n-homogeneous polynomials and $\mathfrak{B}$ and $\mathfrak{C}$ operator ideals. Following [Flo01], we say that a polynomial P is in the **composition ideal** $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ if it admits a factorization $P = S \circ Q \circ T$, with $S \in \mathfrak{C}$, $Q \in \mathfrak{A}_n$ and $T \in \mathfrak{B}$. The ideals being normed, we define the composition quasi-norm

$$\|P\|_{\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}} = \inf\{\|S\|_{\mathfrak{C}} \|Q\|_{\mathfrak{A}_n} \|T\|_{\mathfrak{B}}^n : \text{ all factorizations of } P\}. \tag{1.1}$$

This quasi-norm is actually a λ-norm for some $0 < \lambda \leq 1$ [Flo01]. We say that the composition ideal $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is normed whenever the composition quasi-norm (1.1) is a norm.

A normed ideal of linear operators (polynomials) is closed if the norm considered is the usual linear operator (polynomial) norm. If $\mathfrak{B}$ and $\mathfrak{C}$ are closed operator ideals and $\mathfrak{A}_n$ is normed, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is normed. If $\mathfrak{B}$ is t-normed, $\mathfrak{C}$ is r-normed and $\mathfrak{A}_n$ is s-normed, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is λ-normed, with $\frac{1}{\lambda} = \frac{n}{t} + \frac{1}{s} + \frac{1}{r}$.

For example, $\mathcal{D}_r^n = \mathcal{P}^n \circ \Pi_r$ and $\mathcal{L}_r^n = \mathcal{P}^n \circ \mathcal{L}_r$ (see [Sch91, Flo01]).

Minimal hull

Given a Banach ideal of n-homogeneous polynomials $\mathfrak{A}_n$, the minimal ideal $\mathfrak{A}_n^{\min}$ is defined as

$$\mathfrak{A}_n^{\min} = \overline{\mathcal{F}} \circ \mathfrak{A}_n \circ \overline{\mathcal{F}},$$

where $\overline{\mathcal{F}}$ is the ideal of approximable operators; and

$$\|P\|_{\mathfrak{A}_n^{\min}} = \inf \|S\|_{\overline{\mathcal{F}}} \|Q\|_{\mathfrak{A}_n} \|T\|_{\overline{\mathcal{F}}}^n,$$

where the infimum is taken over all factorizations $P = SQT$ with $S, T \in \overline{\mathcal{F}}$, $Q \in \mathfrak{A}_n$.

Proposition 1.2.1. *[Flo01]*

- $\mathfrak{A}_n^{\min} \subset \mathfrak{A}_n$ *with* $\|\cdot\|_{\mathfrak{A}_n} \leq \|\cdot\|_{\mathfrak{A}_n^{\min}}$.
- $(\mathfrak{A}_n^{\min})^{\min} \overset{1}{=} \mathfrak{A}_n^{\min}$.
- $\mathfrak{A}_n^{\min}$ *is the smallest ideal of n-homogeneous polynomials such that* $\mathfrak{A}_n^{\min}(M, N) \overset{1}{=} \mathfrak{A}_n(M, N)$ *for every finite dimensional Banach spaces M, N.*

- *If E' and F have the metric approximation property, then* $\mathfrak{A}_n^{\min}(E,F) \overset{1}{\hookrightarrow} \mathfrak{A}_n(E,F)$ *and* $\mathfrak{A}_n^{\min}(E,F) \overset{1}{=} \overline{\mathcal{P}_f(E,F)}^{\|\cdot\|_{\mathfrak{A}_n}}$.

A Banach polynomial ideal is minimal if $\mathfrak{A}_n^{\min} = \mathfrak{A}_n$.

For example, the ideals nuclear and approximable polynomials are minimal. Moreover $\mathcal{P}_{PI}^{\min} = \mathcal{P}_{GI}^{\min} = \mathcal{P}_N$ and $\mathcal{P}^{\min} = \mathcal{P}_A$.

Maximal hull

The maximal hull $\mathfrak{A}_n^{\max}$ of a normed polynomial ideal $\mathfrak{A}_n$ is defined as the class of all $P \in \mathcal{P}^n(E,F)$ such that

$$\|P\|_{\mathfrak{A}_n^{\max}(E,F)} := \{\|Q_L^F \circ P \circ J_M^E\|_{\mathfrak{A}_n(M,L)} : \ M \in FIN(E), \ L \in COFIN(F)\} < \infty,$$

where $FIN(E)$ ($COFIN(F)$) denotes the set of finite dimensional (codimensional) subspaces of E (F), and Q_L^F (J_M^E) denote the projection from F onto L (injection from M into E).

$\mathfrak{A}_n^{\max}$ is the largest normed ideal of n-homogeneous polynomials that coincides isometrically with $\mathfrak{A}_n$ in finite dimensional spaces [Flo01, FH02].

A normed polynomial ideal $\mathfrak{A}_n$ is called maximal if $\mathfrak{A}_n^{\max} = \mathfrak{A}_n$.

Proposition 1.2.2. *[Flo01]*

- $\mathfrak{A}_n \subset \mathfrak{A}_n^{\max}$ *with* $\|\cdot\|_{\mathfrak{A}_n^{\max}} \leq \|\cdot\|_{\mathfrak{A}_n}$.
- $(\mathfrak{A}_n^{\max})^{\max} \overset{1}{=} \mathfrak{A}_n^{\max}$.
- $\mathfrak{A}_n^{\max}$ *is the greatest ideal of n-homogeneous polynomials such that* $\mathfrak{A}_n^{\max}(M,N) \overset{1}{=} \mathfrak{A}_n(M,N)$ *for every finite dimensional Banach spaces M, N.*
- $(\mathfrak{A}_n^{\max})^{\min} \overset{1}{=} \mathfrak{A}_n^{\min}$ *and* $(\mathfrak{A}_n^{\min})^{\max} \overset{1}{=} \mathfrak{A}_n^{\max}$.
- *If $\mathfrak{B}$, $\mathfrak{C}$ are maximal ideals of linear operators and $\mathfrak{A}_n$ is maximal, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is maximal.*

For example, $\mathcal{P}, \mathcal{P}_I, \mathcal{P}_e, \mathcal{D}_r, \mathcal{M}_r, \mathcal{P}_r$ are maximal ideals. Also, $\mathcal{P}_N^{\max} = \mathcal{P}_I$ and $\mathcal{P}_A^{\max} = \mathcal{P}$.

1.2.1 Symmetric tensor products

The theory of normed ideals of operators is closely related to the theory of tensor product of Banach spaces. This relationship begun with the work of Grothendieck (see [Gro55]), who defined the projective and injective norms on the (full) 2-fold tensor product of two Banach spaces E and F:

Definition 1.2.3. Let E, F be a Banach space and denote by $E \otimes F$ the (full) 2-fold tensor product of E and F.

1. The *projective tensor norm of order 2*, π, is the norm,

$$\pi(z, E \otimes F) = \inf\{\sum_{j=1}^m \|x_j\|\|y_j\| : \ m \in \mathbb{N}, \ z = \sum_{j=1}^m x_j \otimes y_j\}.$$

2. The *injective tensor norm of order 2*, ε, is the norm,

$$\varepsilon(z, E\otimes F) = \sup\{\Big|\sum_{j=1}^{m}\gamma(x_j)\varphi(y_j)\Big| : \gamma \in B_{E'},\ \varphi \in B_{F'}\},$$

   if $z = \sum_{j=1}^m x_j \otimes y_j$.

We will denote by $E\hat{\otimes}_\pi F$ and $E\hat{\otimes}_\varepsilon F$ the completion of the normed spaces $(E\otimes F, \pi)$ and $(E\otimes F, \varepsilon)$, respectively.

Grothendieck also proved the following theorem, relating the tensor product of Banach spaces with some spaces of operators on E:

Theorem 1.2.4. *$(E\hat{\otimes}_\pi F)' \overset{1}{=} \mathcal{L}(E, F')$ and $(E\hat{\otimes}_\varepsilon F)' \overset{1}{=} \mathcal{L}_{GI}(E, F')$.*

Ryan, in his thesis [Rya80], introduced norms on the symmetric tensor product of Banach spaces to study homogeneous polynomials. The projective and injective tensor norms for the symmetric tensor product are defined as follows:

Definition 1.2.5. Let E be a Banach space and denote by $\bigotimes^{n,s} E$ the n-fold symmetric tensor product of E.

1. The *projective symmetric tensor norm*, π_s is the norm,

$$\pi_s(z, \bigotimes^{n,s} E) = \inf\left\{\sum_{j=1}^{m}|\lambda_j|\|x_j\|^n :\ m\in\mathbb{N},\ z = \sum_{j=1}^{m}\lambda_j x_j^n\right\},$$

   where, $x_j^n = \otimes^n x_j = x_j\otimes\cdots\otimes x_j$.

2. The *injective symmetric tensor norm*, ε_s is the norm,

$$\varepsilon_s(z, \bigotimes^{n,s} E) = \sup\left\{\Big|\sum_{j=1}^{m}\lambda_j\gamma(x_j)^n\Big| : \gamma\in B_{E'}\right\},$$

   if $z = \sum_{j=1}^m \lambda_j x_j^n$.

The following results relate symmetric tensor norms with ideals of polynomials, see [Flo97] ([Vil03] or [CL05] for *4*).

Theorem 1.2.6. *Let E, F be Banach spaces.*

1. *$\mathcal{P}^n(E,F) \overset{1}{=} \mathcal{L}(\hat{\bigotimes}^{n,s}_{\pi_s} E, F)$, where $\hat{\bigotimes}^{n,s}_{\pi_s} E$ denotes the completion of $(\bigotimes^{n,s} E, \pi_s)$. In particular, $\mathcal{P}^n(E) \overset{1}{=} (\hat{\bigotimes}^{n,s}_{\pi_s} E)'$.*

2. *$\mathcal{P}^n(E,F') \overset{1}{=} \big(\hat{\bigotimes}^{n,s}_{\pi_s} E\otimes_\pi F\big)'$, where π denotes the two fold full projective tensor norm.*

3. *If E' has the approximation property, $\hat{\bigotimes}^{n,s}_{\pi_s} E' \overset{1}{=} \mathcal{P}_N(E)$, where we associate $z = \sum_{j=1}^m \lambda_j \otimes^n \gamma_j \in \bigotimes^{n,s} E'$ with the polynomial $P^z \in \mathcal{P}^n(E)$ such that, $P^z(x) = \sum_{j=1}^m \lambda_j\gamma_j(x)^n$ for every $x \in E$.*

4. *$\mathcal{P}^n_{PI}(E,F) \overset{1}{=} \mathcal{L}_{PI}(\hat{\bigotimes}^{n,s}_{\varepsilon_s} E, F)$. The same is true if we replace Pietsch integral by Grothendieck integral mappings. In particular, $\mathcal{P}^n_I(E) \overset{1}{=} (\hat{\bigotimes}^{n,s}_{\varepsilon_s} E)'$.*

5. $\mathcal{P}_A^n(E) \overset{1}{=} \hat{\bigotimes}_{\varepsilon_s}^{n,s} E'$.

More generally, "reasonable" symmetric tensor norms are defined as follows.

Definition 1.2.7. A *symmetric tensor norm of order* n (or just s-tensor norm), α, is an assignment, to each Banach space E of a norm $\alpha(\cdot, \bigotimes^{n,s} E)$ on the n-fold symmetric tensor product $\bigotimes^{n,s} E$ such that
(1) $\varepsilon_s \leq \alpha \leq \pi_s$ on $\bigotimes^{n,s} E$.
(2) α satisfies the metric mapping property, i.e., for every $T \in \mathcal{L}(E, F)$,

$$\| \otimes^{n,s} T \|_{\mathcal{L}(\bigotimes_\alpha^{n,s} E \to \bigotimes_\alpha^{n,s} F)} \leq \|T\|^n_{\mathcal{L}(E,F)}.$$

An s-tensor norm α is called finitely generated if for every E and $z \in \bigotimes^{n,s} E$,

$$\alpha(z, \bigotimes^{n,s} E) = \inf\{\alpha(z, \bigotimes^{n,s} M) : \; M \in FIN(E), \; z \in \bigotimes^{n,s} M\}.$$

For example, π_s and ε_s are finitely generated s-tensor norms, see [Flo97].

Given an s-tensor norm α of order n, we may define a finitely generated s-tensor norm of order n, α', by

$$\bigotimes_{\alpha'}^{n,s} M \overset{1}{=} \Big(\bigotimes_\alpha^{n,s} M' \Big)'$$

for $M \in FIN(M)$. α' is called the *dual* norm of α. It follows that $\pi_s' = \varepsilon_s$ and $\varepsilon_s' = \pi_s$ and that for any finitely generated α, $\alpha'' = \alpha$.

Also, given a scalar normed ideal of n-homogeneous polynomials, $\mathfrak{A}_n$, we can define a finitely generated s-tensor norm α by

$$\bigotimes_\alpha^{n,s} M \overset{1}{=} \mathfrak{A}_n(M'),$$

for $M \in FIN(M)$ and for $z \in \bigotimes^{n,s} E$,

$$\alpha(z; \bigotimes^{n,s} E) := \inf \left\{ \alpha(z; \bigotimes^{n,s} M) : \; M \in FIN(E), \; z \in \bigotimes^{n,s} M \right\}.$$

α is called the *s-tensor norm associated to* $\mathfrak{A}_n$. Note that $\mathfrak{A}_n(M) = \bigotimes_\alpha^{n,s} M' = \big(\bigotimes_{\alpha'}^{n,s} M\big)'$ for every $M \in FIN(M)$.

For example, the s-tensor norm associated to $\mathcal{P}$ and $\mathcal{P}_A$ is ε_s and the s-tensor norm associated to $\mathcal{P}_I$ and $\mathcal{P}_N$ is π_s.

Theorem 1.2.8. *Representation Theorems [Flo01, FH02]*

- *A normed ideal of n-homogeneous polynomials $\mathfrak{A}_n$ is maximal if and only if $\mathfrak{A}_n(E) = (\hat{\bigotimes}_{\alpha'}^{n,s} E)'$, where α is the s-tensor norm associated to $\mathfrak{A}_n$. The norm α' is sometimes called the predual norm to $\mathfrak{A}_n$.*

- *If $\mathfrak{A}_n$ is a Banach ideal of n-homogeneous polynomials with associated s-tensor norm α. Then the natural map*

$$\hat{\bigotimes}_\alpha^{n,s} E' \longrightarrow \mathfrak{A}_n(E),$$

is a metric surjection for every Banach space E, and it is also an isometry if E has the bounded approximation property.

We denote by η (or η_n if we want to specify the order) to the s-tensor norm which is dual to the s-tensor norm associated to the ideal of extendible polynomials. Then, $\mathcal{P}_e^n(E) = (\hat{\bigotimes}_\eta^{n,s} E)'$.

We know recall the definition of adjoint ideal ([Flo01]).

Definition 1.2.9. Let $\mathfrak{A}_n$ be a normed ideal of n-homogeneous polynomials, with associated s-tensor norm α. We may define the *adjoint* (or dual) ideal, $\mathfrak{A}_n^*$, by

$$\mathfrak{A}_n^*(E) \overset{1}{=} (\hat{\bigotimes}_{\alpha}^{n,s} E)'.$$

By the Representation theorem 1.2.8, $\mathfrak{A}_n^*$ is a maximal ideal and $\mathfrak{A}_n^{**} = \mathfrak{A}_n^{\max}$. Also, if E has the bounded approximation property, then $\mathfrak{A}_n(E)' \overset{1}{=} \mathfrak{A}_n^*(E')$.

For example, $\mathcal{P}^* = \mathcal{P}_A^* = \mathcal{P}_I$ and $\mathcal{P}_N^* = \mathcal{P}_I^* = \mathcal{P}$.

The above constructions may be carried out for vector valued ideals also and some representation theorems can be proved, see Sections 2.5 and 2.6, and [Flo01, Section 7].

1.3 Holomorphic functions on Banach spaces

We refer to [Muj86, Din99] for all the material of this section, except the analytic structure of the spectrum which may be found in [AGGM96, DV04]. Let $U \subset E$ be a an open subset. A mapping $f : U \to F$ is *holomorphic* on U if it is continuous and Gateaux-holomorphic on U, that is, for each $\varphi \in F'$, $x_0 \in U$ and $x \in E$, the function $\lambda \mapsto \varphi \circ f(x_0 + \lambda x)$ is holomorphic on some neighbourhood of 0. The set of all holomorphic mappings on U will be denoted $H(U, F)$ (or $H(U)$ if $F = \mathbb{C}$). The following are equivalent:

a. f is holomorphic on U.

b. f is Fréchet differentiable at each point $x_0 \in U$, that is, there exist $T \in \mathcal{L}(E, F)$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - T(x - x_0)}{\|x - x_0\|} = 0$$

c. The Taylor series expansion of f at x_0 converges uniformly on a neighbourhood of each point $x_0 \in U$, that is, there are k-homogeneous polynomials $\frac{d^k f(x_0)}{k!} \in \mathcal{P}^k(E, F)$, $k \geq 0$, such that $f(x) = \sum_{k=0}^{\infty} \frac{d^k f(x_0)}{k!}(x)$.

The polynomial $\frac{d^k f(x_0)}{k!}$ is called the k-differential of f at x_0 and the first differential (or just the differential) of f at x_0, $d^1 f(x_0)$ coincides with the operator T of **b**. The radius of convergence of f at x_0, R, is defined as the supremum of all $r > 0$ such that the Taylor series of f at x_0 converges uniformly on the ball $B(x_0, r)$. The Cauchy-Hadamard formula states that $\frac{1}{R} = \limsup \|\frac{d^k f(x_0)}{k!}\|^{\frac{1}{k}}$.

We also have the Cauchy Integral Formula which states that

$$\frac{d^k f(a)}{k!}(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(a + \lambda x)}{\lambda^{k+1}} d\lambda,$$

where f is a holomorphic function on the open subset U, $a \in U$, $x \in E$ and $r > 0$ is such that $a + \lambda x \in U$ for every $|\lambda| \leq r$. As a corollary we have the Cauchy Inequality: $\left|\frac{d^k f(a)}{k!}(x)\right| \leq \frac{1}{r^k} \sup_{|\lambda|=r} \left|f(a + \lambda x)\right|$. In the case $V \subset U$ is balanced, we have that $\left\|\frac{d^k f(0)}{k!}\right\|_V \leq \|f\|_V$, where, for a function $g : V \to \mathbb{C}$, $\|g\|_V$ denotes the supremum of $|g|$ on V.

A subset $A \subset U$ is U-bounded (denoted by $A \subset\subset U$) if it is bounded and bounded away from the boundary of U (i.e. $dist(A, U^c) > 0$). A function $f : U \to F$ is a *holomorphic function of bounded type* if f maps U-bounded sets to bounded sets of F. For example, the function $g : c_0 \to \mathbb{C}$, $g(x) = \sum_n x_n^n$, where $x = (x_n)_n$, is entire (i.e. holomorphic on E) but it is not bounded in the unit ball of E, thus it is not a bounded type function on E. It is a consequence of a theorem of Josefson and Nissenzweig that on every infinite dimensional Banach space there are holomorphic functions which are not of bounded type.

The space of holomorphic functions of bounded type on U is denoted by $H_b(U,F)$ (or $H_b(U)$ if $F=\mathbb{C}$). Let $U_n=\{x\in U:\ \|x\|\le n,\ dist(x,U^c)\ge \frac{1}{n}\}$. Then we may define on $H_b(U,F)$, the seminorms $q_n(f)=\|f\|_{U_n}:=\sup\{|f(x)|:\ x\in U_n\}$. The space $(H_b(U,F),(q_n)_n)$ is a Fréchet space. In case $F=\mathbb{C}$, $H_b(U)$ is a locally m-convex Fréchet algebra[1].

It is known that for balanced open sets U, polynomials are dense in $H_b(U)$ (see for example [Muj86, Theorem 7.11]).

The entire functions of bounded type may be characterized as the entire functions such that $\|\frac{d^kf(x_0)}{k!}\|^{\frac{1}{k}}\to 0$ as $k\to\infty$. In this case the topology of $H_b(E,F)$ may be described with the seminorms $p_r(f):=\sum_{k=0}^{\infty} r^k\|\frac{d^kf(x_0)}{k!}\|^{\frac{1}{k}}$, $r>0$.

A Riemann domain (X,p) spread over a Banach space E is a Hausdorff topological space X and a local homeomorphism $p:X\to E$. If $r>0$, $x\in X$ and there exists a neighborhood V of x such that $p_{|_V}$ is an homeomorphism onto $B_r(p(x))$, then V is denoted as $B_r(x)$. The distance of a point $x\in X$ to the boundary is defined as the $dist_X(x)=\sup\{r>0:\ B_r(x) \text{ exists}\}$. A function $f:X\to\mathbb{C}$ is *holomorphic* on X if for each $x\in X$ and $r<dist_X(x)$, the function $f\circ(p|_{B_r(x)})^{-1}$ is holomorphic. For $x\in X$ and $r<dist_X(x)$, we define $\frac{d^kf(x)}{k!}=\frac{d^k(f\circ(p|_{B_r(x)})^{-1})(x)}{k!}$.

A subset $A\subset X$ is X-bounded if $p(A)$ is bounded and $dist_X(A)=\inf\{dist_X(x):\ x\in A\}$ is positive. $f:X\to\mathbb{C}$ is holomorphic of bounded type on X if it is holomorphic and it is bounded on each X-bounded subset. The set of all holomorphic functions of bounded type on X is denoted by $H_b(X)$. The space $H_b(X)$ is a Fréchet algebra when it is considered with the topology of uniform convergence on X-bounded sets.

1.3.1 The Aron-Berner extension

There is a natural way to extend linear functionals on E to w^*-continuous linear functionals on the bidual E''. Aron and Berner [AB78] showed that this extension may be carried out also for homogeneous polynomials and holomorphic functions of bounded type.

Let $A\in\mathcal{L}^n_s(E)$ be a symmetric n-linear form. The Aron-Berner extension of A, $AB(A)$ is an n-linear form on E''. For $x_1,\dots,x_{n-1}\in E$, note that $A(x_1,\dots,x_{n-1},\cdot)$ belongs to E'. Thus for each $z\in E''$, we may define $\overline{z}:\mathcal{L}^n_s(E)\to\mathcal{L}^{n-1}_s(E)$ by,

$$\overline{z}(A)(x_1,\dots,x_{n-1})=z(A(x_1,\dots,x_{n-1},\cdot)).$$

Similarly, for each $1\le k<n$, we can define $\overline{z}:\mathcal{L}^k_s(E)\to\mathcal{L}^{k-1}_s(E)$. Thus the Aron-Berner extension of A is defined by

$$AB(A)(z_1,\dots,z_n)=\overline{z}_1\circ\cdots\circ\overline{z}_n(A).$$

The Aron-Berner extension is not, in general, symmetric. Moreover, we have chosen an order to pick the variables of A, and in general, the extension obtained depends on this order. However, it has the following properties:

- If $x\in E$ and $z_1,\dots,z_{n-1}\in E''$ then

$$AB(A)(x,z_1,\dots,z_{n-1})=AB(A)(z_1,x,\dots,z_{n-1})=\cdots=AB(A)(z_1,\dots,z_{n-1},x).$$

[1] A locally m-convex Fréchet algebra, or just Fréchet algebra, Y, is a Fréchet space which is an algebra and whose topology may be given by seminorms q which are submultiplicative, i.e. $q(xy)\le q(x)q(y)$ for every $x,y\in Y$.

- It is w^*-w^*-continuous in the first variable (the last variable which is extended).
- If $(x_{\alpha_k})_{\alpha_k} \subset E$ are nets converging to $z_k \in E''$, $k = 1, \dots, n$, then
$$AB(A)(z_1, \dots, z_n) = \lim_{\alpha_1} \dots \lim_{\alpha_n} A(x_{\alpha_1}, \dots, x_{\alpha_n}).$$
- $\|AB(A)\| = \|A\|$.
- $AB(A)$ is separately w^*-continuous on each variable if and only if $AB(A)$ is symmetric.

A Banach space E is *symmetrically regular* if the Aron-Berner extension of each symmetric multilinear form is symmetric. For example, any reflexive Banach space, ℓ_∞ and c_0 are symmetrically regular; ℓ_1 is not symmetrically regular.

The restriction of $AB(A)$ to the diagonal is unique, that is, the application $E'' \ni z \mapsto AB(A)(z, \dots, z)$ does not depend on the order of the variables chosen to extend A. Thus if $P \in \mathcal{P}^n(E, F)$, then its Aron-Berner extension $AB(P) \in \mathcal{P}^n(E'', F'')$ is uniquely defined as $AB(P)(z) := AB(\overset{\vee}{P})(z, \dots, z)$. Davie and Gamelin proved that the Aron-Berner extension $AB : \mathcal{P}^n(E) \to \mathcal{P}^n(E'')$ is an isometry [DG89]. Moreover, they extended Goldstine's theorem proving that B_E is polynomial-star dense in $B_{E''}$, that is, for each $z \in B_{E''}$ there exists a net $(x_\alpha)_\alpha \subset B_E$, such that $P(x_\alpha) \to P(z)$ for every polynomial P.

Given a holomorphic function f on E, we may extend each of the homogeneous polynomials in the Taylor series of f to obtain a holomorphic function on some neighbourhood of E''. This procedure works fine for functions of bounded type:

- The Aron-Berner extension induces a continuous and multiplicative homomorphism $AB : H_b(E) \to H_b(E'')$.

In contrast we have the following result: a holomorphic function f on c_0 is extendible to a holomorphic function on ℓ_∞ if and only if f belongs to $H_b(c_0)$.

The spectrum of H_b.

Let (X, p) be a Riemann domain over E. We will denote by $M_b(X)$ the spectrum of the algebra $H_b(X)$, that is, the set of all non-zero continuous, linear and multiplicative functionals on $H_b(X)$. Thus, for each $\varphi \in M_b(X)$ there exists an X-bounded set B such that $\phi(f) \leq \sup_{x \in B} |f(x)|$, for all $f \in H_b(X)$. In this case, we will write $\varphi \prec B$. By a *fundamental sequence of X-bounded sets* we will mean a sequence $\{A_n\}_n$ of X-bounded subsets such that if B is another X-bounded subset, then there exists n_0 such that $B \subset A_{n_0}$. We denote $X_r := \{x \in X : dist_X(x) \geq \frac{1}{r} \text{ and } \|p(x)\| \leq r\}$. Note that $\{X_n\}_{n \in \mathbb{N}}$ is a fundamental sequence of X-bounded sets.

For the case $X = E$, we can define an application $\pi : M_b(E) \to E''$ by $\pi(\varphi) = \varphi_{|_{E'}}$. By the Aron-Berner extension, the evaluations δ_z at points $z \in E''$ are continuous homomorphisms, and thus, E'' is identified with a part of $M_b(E)$. Note that $\pi(\delta_z) = z$ for every $z \in E''$, therefore π is surjective. In general, π is not injective. Indeed, the following holds:

- If there exists a polynomial P which is not weakly continuous on bounded sets then there exists $\varphi \in M_b(E)$ such that $\varphi \neq \delta_{\pi(\varphi)}$.

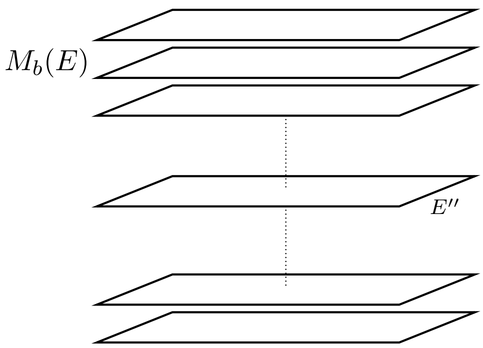
For a general Riemann domain (X, p), the mapping $\pi : M_b(X) \to E''$ is defined by $\pi(\varphi)(\gamma) = \varphi(\gamma \circ p)$. If E is symmetrically regular, this mapping π provides the local homeomorphism that makes $M_b(X)$ a Riemann domain over E'' [AGGM96, DV04]. We briefly describe their construction. Given $f \in H_b(X)$, $z \in E''$, the function $X \ni x \mapsto AB(\frac{d^k f(x)}{k!})(z)$ is holomorphic of bounded type. For $\varphi \in M_b(E)$ such that $\varphi \prec X_n$ and $z \in E''$, with $\|z\| < \frac{1}{n}$, we may define $\varphi^z \in M_b(X)$ by[2]

$$\varphi^z(f) = \sum_{k=0}^{\infty} \varphi\Big(x \mapsto AB(\frac{d^k f(x)}{k!})(z)\Big).$$

If E is symmetrically regular, the Aron-Berner extensions of symmetric multilinear mappings are symmetric, and this allows to prove that the sets $V_{\varphi,r} := \{\varphi^z : \|z\| < r,\} \subset M_b(E)$, $\varphi \in M_b(X)$, $\varphi \prec X_r$, constitute a neighbourhood basis for a Hausdorff topology on $M_b(X)$. Moreover, $\sup\{dist_X(A) : \varphi \prec A\} \leq dist_{M_b(X)}(\varphi)$. It is also proved $\pi(\varphi^z) = \pi(\varphi) + z$, and from this it is easy to see that π is a local homeomorphism and therefore we have:

Theorem 1.3.1. *[AGGM96, Corollary 2.4] and [DV04, Propositons 1.5 and 2.3] Let (X,p) is a Riemann domain over a symmetrically regular Banach space E. Then $(M_b(X), \pi)$ is a Riemann domain over E''. Moreover, every bounded type holomorphic function f on X extend to a holomorphic function $\tilde{f}$ on $M_b(X)$ via its Gelfand transform (i.e. $\tilde{f}(\varphi) = \varphi(f)$).*

In the case $X = E$, the connected component containing $\varphi \in M_b(E)$ is the *sheet* of φ, $S(\varphi) = \{\varphi^z : z \in E''\}$. Therefore each connected component of $M_b(E)$ is homeomorphic to E'':



It was shown in [Din99, Section 6.3] that the extensions of bounded type entire functions on E to the spectrum are of bounded type on each connected component of $M_b(E)$ (see Proposition 5.5.2). In contrast, if E is not symmetrically regular, the sets $V_{\varphi,r}$ do not even define a topology in $M_b(E)$.

[2] For $X = E$ we may simply define $\varphi^z(f) = \varphi(f(z + \cdot))$, see [Din99, Section 6.3].

Chapter 2

Compatible polynomial ideals on Banach spaces

In this chapter, in order to investigate the relationship between an operator ideal and its natural polynomial extensions, we define the concept of compatibility. We study the stability of these properties for maximal and minimal hulls, adjoint and composition ideals. We also relate these concepts with conditions on the underlying tensor norms and with interpolation spaces. The content of this chapter appears in [CDM09].

2.1 Definitions and general results

Many examples of polynomial ideals appear as generalizations of ideals of operators. We intend to clarify the relationship between an ideal of operators and its possible generalization to higher degrees. In particular, we are interested in properties that are shared by the operator and polynomial ideals.

Next lemma shows that any polynomial ideal is closed by the combined operation of fixing variables followed by multiplication by a power of a linear functional.

Lemma 2.1.1. *Let $\mathfrak{A}_n$ be an ideal of n-homogeneous polynomials and $P \in \mathfrak{A}_n(E, F)$. If $T_1, \dots, T_n \in \mathcal{L}(G, E)$, then the n-homogeneous polynomial given by $Q(\cdot) = \overset{\vee}{P}(T_1(\cdot), \dots, T_n(\cdot))$ belongs to $\mathfrak{A}_n(G, F)$.*

Moreover, if $0 < j < n$, $\gamma \in E'$, and $a \in E$, then

(a) $\gamma^j P_{a^j}$ belongs to $\mathfrak{A}_n(E, F)$.

(b) $\left(\gamma^j P\right)_{a^j}$ belongs to $\mathfrak{A}_n(E, F)$.

Proof. The first assertion follows from the polarization formula. Statement (a) follows from this fact and the equality

$$\gamma(x)^j P_{a^j}(x) = \overset{\vee}{P}(\gamma(x)a, \dots, \gamma(x)a, x \dots, x).$$

To prove (b), we expand $\left(\gamma^j P\right)_{a^j}$ as

$$\begin{aligned}
\left(\gamma^j P\right)_{a^j}(x) &= \frac{1}{\binom{n+j}{j}} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{j-i} \gamma(a)^i \gamma(x)^{j-i} \overset{\vee}{P}(a^{j-i}, x^{n-j+i}) \\
&= \frac{1}{\binom{n+j}{j}} \sum_{i=0}^{j} \binom{j}{i} \binom{n}{j-i} \gamma(a)^i \left(\gamma^{j-i}\ P_{a^{j-i}}\right)(x),
\end{aligned}$$

and use (a). □

This result suggests that the operations of fixing a variable or multiplying by a linear functional are inherent to the structure of polynomial ideals. These natural operations have been considered by several authors to relate spaces of polynomials of different degrees. In particular, the operation of fixing a variable is intrinsic to the definition of holomorphy type [Nac69] (see also [Din71a, BBJP06]). It also motivated the definition of ideal of polynomials "closed under differentiation" [BP05] and the polynomial property (B) [BBJP06]. On the other hand, the operation of multiplying by a linear functional originated the concept of ideal of polynomials "closed for scalar multiplication" introduced in [BP05]. Our purpose is to relate ideals of polynomials with ideals of operators in this chapter and ideals of polynomials of different degrees in the next one. In both cases, this can be done by the natural operations mentioned above. The following definition consider the joint effect of both operations with control of the ideal norms.

Definition 2.1.2. *Let $\mathfrak{A}$ be a quasi-normed ideal of linear operators. We say that the quasi-normed ideal of n-homogeneous polynomials $\mathfrak{A}_n$ is* ***compatible with*** *$\mathfrak{A}$ (or that $\mathfrak{A}_n$ and $\mathfrak{A}$ are compatible) if there exist positive constants A and B such that for every Banach spaces E and F, the following conditions hold:*
(i) For each $P \in \mathfrak{A}_n(E,F)$ and $a \in E$, $P_{a^{n-1}}$ belongs to $\mathfrak{A}(E;F)$ and

$$\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq A\|P\|_{\mathfrak{A}_n(E,F)}\|a\|^{n-1}$$

(ii) For each $T \in \mathfrak{A}(E,F)$ and $\gamma \in E'$, $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E,F)$ and

$$\|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)} \leq B\|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E,F)}$$

We will sometimes write $\mathfrak{A}_n \sim \mathfrak{A}$ to denote that $\mathfrak{A}$ and $\mathfrak{A}_n$ are compatible.

Note that we could have defined analogously a notion of compatibility between two (or more) quasi-normed ideals of homogeneous polynomials of different degrees. However we will restrict ourselves to compare an operator ideal with an ideal of polynomials in this chapter and in the next one a full sequence of k-homogeneous polynomial ideals $\{\mathfrak{A}_k\}_{k\in\mathbb{N}}$.

Although the definition of compatibility involves constants which relate the norms of the operators and the homogeneous polynomials, when the ideals are complete those constants automatically exist. This means that if we can define the operations of fixing variables and multiplying by functionals, then they are uniformly (in the Banach spaces E, F) bounded. This is proved in the next result.

Proposition 2.1.3. *Let $\mathfrak{A}_n, \mathfrak{A}$ be Banach ideals of n-homogeneous polynomials and linear operators respectively. Suppose that for every Banach spaces E, F,*

(a) if $a \in E$ and $P \in \mathfrak{A}_n(E,F)$ then $P_{a^{n-1}} \in \mathfrak{A}(E,F)$, and

(b) if $\gamma \in E'$ and $T \in \mathfrak{A}(E,F)$ then $\gamma^{n-1}T \in \mathfrak{A}_n(E,F)$.

Then $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$, that is, the norm conditions in Definition 2.1.2 are automatically satisfied.

Proof. Let us first prove that there exists a constant $A > 0$ (independent of the spaces E, F) such that, for every $a \in E$ and $P \in \mathfrak{A}_n(E,F)$, $\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq A\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)}$. We will prove this proposition in three steps:

(1) For fixed E, F and $a \in E$, the application

$$\begin{array}{rccc} \phi_{a^{n-1}}: & \mathfrak{A}_n(E,F) & \to & \mathfrak{A}(E,F) \\ & P & \mapsto & P_{a^{n-1}}, \end{array}$$

is continuous.

Proof. Just apply the Closed Graph Theorem.

(3) For fixed E, F the mapping,

$$\begin{array}{rccc} \phi: & E & \to & \mathcal{L}(\mathfrak{A}_n(E,F), \mathfrak{A}(E,F)) \\ & a & \mapsto & \phi_{a^{n-1}}(P) = P_{a^{n-1}}, \end{array}$$

is continuous. Thus here exists a constant $A = A_{E,F} > 0$ such that, for every $a \in E$ and $P \in \mathfrak{A}_n(E,F)$, $\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq A\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)}$.

Proof. By (1), the application ϕ is well defined. Step (2) then follows from the Multilinear Closed Graph Theorem.

(2) There exists a constant $A > 0$ (independent of the spaces E, F) such that, for every $a \in E$ and $P \in \mathfrak{A}_n(E,F)$, $\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq A\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)}$.

Proof. Suppose that there exist Banach spaces E_k, F_k and $a_k \in E_k$, with $\|a_k\| < \frac{1}{2^k}$ and $\|\phi_{a_k^{n-1}}\| > k$, where

$$\begin{array}{rccc} \phi_{a_k^{n-1}}: & \mathfrak{A}_n(E_k,F_k) & \to & \mathfrak{A}(E_k,F_k) \\ & P & \mapsto & P_{a_k^{n-1}}. \end{array}$$

Let $P_k \in \mathfrak{A}_k(E_k, F_k)$ such that $\|\phi_{a_k^{n-1}}(P_k)\| > k\|P_k\|_{\mathfrak{A}_n(E_k,F_k)}$. Define the spaces $E = \bigoplus_k E_k$ and $F = \bigoplus_k F_k$ normed in any way such that the applications

$$\begin{array}{ccccc} E_k & \overset{i_k}{\hookrightarrow} & E & \overset{\pi_k}{\to} & E_k \\ F_k & \overset{\tilde{i}_k}{\hookrightarrow} & F & \overset{\tilde{\pi}_k}{\to} & F_k, \end{array}$$

have norm one. Let Q_k be the polynomials $Q_k = \tilde{i}_k \circ P_k \circ \pi_k \in \mathfrak{A}_n(E,F)$. Since $\|a_k\| < \frac{1}{2^k}$, we may define $a = \sum_k a_k \in E$ and also $\phi_{a^{n-1}} : \mathfrak{A}_n(E,F) \to \mathfrak{A}(E,F)$. Note that for every $x \in E$,

$$\phi_{a^{n-1}}(Q_k)(x) = \overset{\vee}{Q}_k(a^{n-1}, x) = \tilde{i}_k \circ \overset{\vee}{P}_k(\pi_k(a)^{n-1}, \pi_k(x)) = \big(\tilde{i}_k \circ (P_k)_{a^{n-1}} \circ \pi_k\big)(x).$$

Thus,

$$\begin{aligned} \|\phi_{a^{n-1}}(Q_k)\|_{\mathfrak{A}(E,F)} &= \|\tilde{i}_k \circ (P_k)_{a^{n-1}} \circ \pi_k\|_{\mathfrak{A}(E,F)} \geq \|\tilde{\pi}_k \circ \tilde{i}_k \circ (P_k)_{a^{n-1}} \circ \pi_k \circ i_k\|_{\mathfrak{A}(E_k,F_k)} \\ &= \|(P_k)_{a^{n-1}}\|_{\mathfrak{A}(E_k,F_k)} > k. \end{aligned}$$

Therefore $\phi_{a^{n-1}}$ cannot be continuous, which contradicts (1).

The fact that there exist a constant $B > 0$ (independent of the Banach spaces E, F) such that for every $\gamma \in E'$ and every $T \in \mathfrak{A}(E,F)$, $\|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)} \leq B\|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E,F)}$ can be proved analogously. □

Even though it is not necessary to obtain the constants A and B to show that two Banach ideals are compatible, we will also seek "good" constants mostly for two reasons: the first one is that this kind of bounds will allow us in the next chapter to define holomorphic mappings associated to sequences of ideals and the second is that they provide a bound for the norm of the derivatives of homogeneous polynomials in different ideals.

The following lemma show a kind of converse to conditions *(i)* and *(ii)* of Definition 2.1.2.

Lemma 2.1.4. *Let $\mathfrak{A}_n$ be an ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$ and $T \in \mathcal{L}(E, F)$. Then the following are equivalent:*

a) $T \in \mathfrak{A}(E, F)$.

b) $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ for all $\gamma \in E'$.

c) $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ for some nonzero $\gamma \in E'$.

d) There exist $P \in \mathfrak{A}_n(E, F)$ and $a \in E$ such that $T = P_{a^{n-1}}$.

e) For each $0 \neq a \in E$, there exist $P \in \mathfrak{A}_n(E, F)$ such that $T = P_{a^{n-1}}$.

Proof. The definition of compatibility implies that $a) \Rightarrow b)$, $a) \Rightarrow c)$, $d) \Rightarrow a)$ and $e) \Rightarrow a)$. Clearly $b) \Rightarrow c)$ and $e) \Rightarrow d)$. So it sufices to prove that $c) \Rightarrow a)$ and that $a) \Rightarrow e)$.

$c) \Rightarrow a)$: Suppose that $Q = \gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E, F)$ and choose $a \in E$ such that $\gamma(a) \neq 0$. Then $Q_{a^{n-1}} \in \mathfrak{A}(E, F)$, since $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$. Now,

$$Q_{a^{n-1}} = \frac{1}{n}\gamma^{n-1}(a)T + \frac{n-1}{n}\gamma^{n-2}(a)T(a)\gamma.$$

So, we can express T as

$$T = \frac{n}{\gamma^{n-1}(a)}Q_{a^{n-1}} - \frac{n-1}{\gamma(a)}T(a)\gamma. \tag{2.1}$$

Since T is a linear combination of a finite type operator and $Q_{a^{n-1}}$, we conclude that it belongs to $\mathfrak{A}(E, F)$.

$a) \Rightarrow e)$: Take $T \in \mathfrak{A}(E, F)$ and $0 \neq a \in E$ and choose $\gamma \in E'$ such that $\gamma(a) = 1$. By equation (2.1), T can be written as

$$T = \Big(n(\gamma^{n-1}T) - (n-1)T(a)\gamma^n\Big)_{a^{n-1}}$$

and the polynomial $P = n(\gamma^{n-1}T) - (n-1)T(a)\gamma^n$ belongs to $\mathfrak{A}_n(E, F)$. $\square$

The previous lemma allows to infer relationships between operator ideals from compatible polynomial ideals.

Proposition 2.1.5. *Let $\mathfrak{A}_n^1, \dots, \mathfrak{A}_n^k, \mathfrak{B}_n$ be quasi-normed ideals of n-homogeneous polynomials compatible with $\mathfrak{A}^1, \dots, \mathfrak{A}^k, \mathfrak{B}$ respectively. If for some E and F, $\bigcap_j \mathfrak{A}_n^j(E, F) \subset \mathfrak{B}_n(E, F)$, then $\bigcap_j \mathfrak{A}^j(E, F) \subset \mathfrak{B}(E, F)$.*

Proof. For $u \in \bigcap_j \mathfrak{A}^j(E, F)$ and a nonzero $\gamma \in E'$, we have $\gamma^{n-1}u \in \bigcap_j \mathfrak{A}_n^j(E, F)$. Thus, $\gamma^{n-1}u \in \mathfrak{B}_n(E, F)$, and by Lemma 2.1.4, $u \in \mathfrak{B}(E, F)$. $\square$

The above proposition was proved, with a different terminology, by Botelho and Pellegrino [BP05, Proposition 2] (see also [BBJP06]).

Note that we only need that $\mathfrak{A}$ satisfies (ii) and $\mathfrak{B}$ satisfies (i) in the definition to obtain the conclusions of the proposition (in fact, none of the norm inequalities in (i) and (ii) are necessary).

The previous proposition asserts that an ideal of n-homogeneous polynomials $\mathfrak{A}_n$ can be compatible with at most one operator ideal $\mathfrak{A}$. We will see later that given an operator ideal there are always many polynomial ideals compatible with it.

Before presenting examples, we need some technical results that will be frequently used throughout this work.

Let $\sigma\colon \bigotimes^n E \to \bigotimes^{n,s} E$ be the symmetrization operator

$$\sigma(x_1 \otimes \cdots \otimes x_n) = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)},$$

where S_n denotes the set of all permutations of $\{1, \ldots, n\}$.

The following result can be derived from [Har97, Corollary 3]. However, we provide a simple proof for the sake of completeness.

Lemma 2.1.6. *Let $\sigma\colon \bigotimes^n E \to \bigotimes^{n,s} E$ be the symmetrization operator. Then, for any symmetric n-tensor norm α and all $a, b \in E$ we have*

$$\alpha\Big(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\Big) \leq e\|a\|\|b\|^{n-1}.$$

Proof. Let $r \in \mathbb{C}$ be a primary root of the unit: $r^n = 1$ and $r^j \neq 1$ for every $1 \leq j < n$. Let us see that for all $t > 0$

$$\sigma(a \otimes b \otimes \cdots \otimes b) = \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \big(a + \frac{r^j}{t} b\big)^n.$$

Indeed, if $P \in \mathcal{P}^n(E)$, $\langle P, \sigma(a \otimes b \otimes \cdots \otimes b)\rangle = \overset{\vee}{P}(a, b, \cdots, b)$, and

$$\begin{aligned}
\Big\langle P, \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \big(a + \frac{r^j}{t} b\big)^n \Big\rangle &= \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} P\big(a + \frac{r^j}{t} b\big) \\
&= \frac{1}{n^2} \sum_{j=0}^{n-1} r^j t^{n-1} \sum_{i=0}^{n} \binom{n}{i} \overset{\vee}{P}\Big(a^{n-i}, \big(\frac{r^j}{t} b\big)^i\Big) \\
&= \frac{t^{n-1}}{n^2} \sum_{i=0}^{n} \binom{n}{i} \frac{1}{t^i} \overset{\vee}{P}(a^{n-i}, b^i) \sum_{j=0}^{n-1} r^{j(i+1)} \\
&= \frac{t^{n-1}}{n^2} \binom{n}{n-1} \frac{1}{t^{n-1}} \overset{\vee}{P}(a, b^{n-1}) n \\
&= \overset{\vee}{P}(a, b, \cdots, b).
\end{aligned}$$

Suppose that $\|a\| = \|b\| = 1$. Then for all $t > 0$ we have that

$$\begin{aligned}
\alpha\Big(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\Big) &\leq \frac{1}{n^2} \sum_{j=0}^{n-1} t^{n-1} \|\frac{r^j}{t} b + a\|^n \\
&\leq \frac{1}{n^2} \sum_{j=0}^{n-1} t^{n-1} \big(\frac{1}{t} + 1\big)^n = \frac{1}{n} t^{n-1} \big(\frac{1}{t} + 1\big)^n.
\end{aligned}$$

Choosing $t = \frac{1}{n-1}$ we obtain

$$\alpha\Big(\sigma(a \otimes b \otimes \cdots \otimes b); \bigotimes^{n,s} E\Big) \leq \left(\frac{n}{n-1}\right)^{n-1} \leq e.$$

Thus for all $a, b \in E$,

$$\begin{aligned}\alpha\Big(\sigma(a \otimes b \otimes \cdots \otimes b)\Big) &= \|a\|\|b\|^{n-1}\alpha\Big(\sigma\big(\frac{a}{\|a\|} \otimes \frac{b}{\|b\|} \otimes \cdots \otimes \frac{b}{\|b\|}\big)\Big) \\ &\leq e\|a\|\|b\|^{n-1}.\end{aligned}$$

□

From the previous proof we obtain the useful expression

$$\sigma(a \otimes b \otimes \cdots \otimes b) = \frac{1}{n^2}\frac{1}{(n-1)^{n-1}}\sum_{j=0}^{n-1} r^j\big((n-1)r^j b + a\big)^n. \tag{2.2}$$

Corollary 2.1.7. *a) For any normed ideal $\mathfrak{A}_n$ of n-homogeneous polynomials, $\gamma, \phi \in E'$ and $y \in F$, we have*

$$\|\gamma\phi^{n-1}y\|_{\mathfrak{A}_n(E,F)} \leq e\|\gamma\|\|\phi\|^{n-1}\|y\|.$$

b) Let $P \in \mathcal{P}^n(E, F)$, and $a, b \in E$. Then

$$\overset{\vee}{P}(a, b^{n-1}) = \frac{1}{n^2}\frac{1}{(n-1)^{n-1}}\sum_{j=0}^{n-1} r^j P\big((n-1)r^j b + a\big)$$

and

$$\|\overset{\vee}{P}(a, b^{n-1})\| \leq e\|P\|\|a\|\|b\|^{n-1}.$$

Proof. $a)$ Define $T \in \mathcal{L}(\mathbb{C}, F)$ as $T(c) = cy$. Then

$$\|\gamma\phi^{n-1}y\|_{\mathfrak{A}_n(E,F)} \leq \|\gamma\phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})}\|T\|_{\mathcal{L}(\mathbb{C},F)} = \|\gamma\phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})}\|y\|_{\mathcal{L}(\mathbb{C},F)}$$

Since we always have the norm one inclusion $\bigotimes_{\pi_s}^{n,s} E' \hookrightarrow \mathfrak{A}_n(E, \mathbb{C})$, we obtain $\|\gamma\phi^{n-1}\|_{\mathfrak{A}_n(E,\mathbb{C})} \leq \pi_s\big(\sigma(\gamma \otimes \phi \otimes \cdots \otimes \phi); \bigotimes^{n,s} E'\big) \leq e\|\gamma\|\|\phi\|^{n-1}$, which ends the proof.

$b)$ This follows from expression (2.2) and the previous lemma. □

Now we show that most classical ideals of polynomials are compatible with the operator ideal generally associated to it.

Example 2.1.8. *Continuous homogeneous polynomials: $\mathcal{P}^n \sim \mathcal{L}$.*

As a consequence of the previous corollary, for each n, $\mathcal{P}^n$ is compatible with the ideal $\mathcal{L}$ of all continuous linear operators with constants $A = e$ and $B = 1$.

Similarly, the ideals of *approximable, compact, weakly compact, weakly sequentially continuous* and *weakly continuous on bounded sets* n-homogeneous polynomials are compatible with the corresponding operator ideals, with constants $A = e$ and $B = 1$.

Example 2.1.9. *Nuclear polynomials: $\mathcal{P}_N^n \sim \mathcal{L}_N$*

Let $P \in \mathcal{P}_N^n(E, F)$. For $a \in E$, it is immediate that $P_{a^{n-1}}$ is nuclear and $\|P_{a^{n-1}}\|_{\mathcal{P}_N^{k-1}(E;F)} \leq \|a\|^{n-1}\|P\|_{\mathcal{P}_N^k(E;F)}$. Also, by Corollary 2.1.7 $a)$, we have that if $T \in \mathcal{L}_N(E, F)$ is a nuclear operator then $\|\gamma^{n-1}T\|_{\mathcal{P}_N^{k+1}(E;F)} \leq e\|\gamma\|^{n-1}\|T\|_{\mathcal{P}_N^k(E;F)}$ for any $\gamma \in E'$. Therefore, the ideal of nuclear polynomials is compatible with the ideal of nuclear operators with constants $A = 1$ and $B = e$.

Example 2.1.10. *Integral polynomials: $\mathcal{P}^n_{PI} \sim \mathcal{L}_{PI}$ and $\mathcal{P}^n_{GI} \sim \mathcal{L}_{GI}$*

The integral polynomials are compatible with the ideal of integral operators, with constants $A = 1$ and $B = e$, as we will see in Sections 2.5 and 2.6.

Example 2.1.11. *Extendible polynomials: $\mathcal{P}^n_e \sim \mathcal{L}_e$*

The sequence of extendible polynomials is compatible with the ideal of extendible operators, with constants $A = e$ and $B = 1$:

(i) Take $P \in \mathcal{P}^n_e(E, F)$ and $a \in E$, suppose $E \subset G$. If $\widetilde{P} \in \mathcal{P}^n(G, F)$ is any extension of P to G, by the polarization formula, $(\widetilde{P})_{a^{n-1}}$ is an extension of $P_{a^{n-1}}$ to G with norm $\|(\widetilde{P})_{a^{n-1}}\| \leq e\|\widetilde{P}\|\|a\|^{n-1}$ (by Corollary 2.1.7 b)). This implies that $P_{a^{n-1}}$ is extendible and $\|P_{a^{n-1}}\|_{\mathcal{L}_e} \leq e\|P\|_{\mathcal{P}^n_e}\|a\|^{n-1}$.

(ii) Let $T \in \mathcal{L}^n_e(E, F)$ and $\gamma \in E'$. If $\widetilde{\gamma}$ is an extension of γ and $\widetilde{T}$ an extension of T to $G \supset E$, then $\widetilde{\gamma}^{n-1}\widetilde{T}$ is an extension of $\gamma^{n-1}T$ to G with norm at most $\|\gamma\|^{n-1}\|T\|$. Thus $\gamma^{n-1}T \in \mathcal{P}^n_e(E, F)$ and $\|\gamma^{n-1}T\|_{\mathcal{P}^n_e} \leq \|\gamma\|^{n-1}\|T\|_{\mathcal{P}^n_e}$.

Example 2.1.12. *Multiple r-summing polynomials: $\mathcal{M}^n_r \sim \Pi_r$*

We will prove that $\mathcal{M}^n_r$ is compatible with the ideal of absolutely r-summing operators with constants $A = B = 1$ as an immediate consequence of Example 3.1.9.

Example 2.1.13. *r-dominated polynomials: $\mathcal{D}^n_r \sim \Pi_r$*

The ideals of r-dominated polynomials are compatible with the ideal of absolutely r-summing operators with constants $A = e$ and $B = 1$. This is a particular case of the composition ideals considered in Section 2.3.

We will see a lot more of examples of compatible ideals later in this chapter. Now we see that not all the usual polynomial extensions of an operator ideal are compatible.

Example 2.1.14. *The ideal of absolutely 1-summing polynomials is not compatible with the ideal of absolutely 1-summing operators: $\Pi^n_1 \nsim \Pi_1$.*

We show that the ideal of absolutely 1-summing 2-homogeneous polynomials is not a compatible extension of the ideal of absolutely 1-summing operators, exhibiting a 2-homogeneous absolutely 1-summing polynomial that does not satisfy condition (i).

Let $P\colon \ell_2 \to \ell_2 \otimes_\pi \ell_2$ be the polynomial given by $P(x) = x \otimes x$. Suppose that $(x_k)_k$ is a weakly 1-summing sequence in ℓ_2, then by Orlicz Theorem (see [DJT95, Theorem 3.12]), $(x_k)_k$ is strongly 2-summing. Therefore

$$\sum \|P(x_k)\| = \sum \pi(x_k \otimes x_k, \ell_2 \otimes \ell_2) = \sum \|x_k\|^2 < \infty$$

which means that P is absolutely 1-summing. On the other hand, let $(e_n)_n$ denote the canonic basis in ℓ_2. Let us see that P_{e_1} is not absolutely 1-summing. Note that $P_{e_1}(e_n) = \frac{e_1 \otimes e_n + e_n \otimes e_1}{2}$ does not converge to 0 in $\ell_2 \otimes \ell_2$-norm. To see this, take $e'_1 + e'_n \in \ell'_2$, then $\|(e'_1 + e'_n)^2\|_{\mathcal{P}^2(\ell_2)} = 2$ and

$$\begin{aligned}
\pi(P_{e_1}(e_n), \ell_2 \otimes \ell_2) &= \sup\{|Q(P_{e_1}(e_n))| : Q \in \mathcal{P}^2(\ell_2), \|Q\| \leq 1\} \geq \frac{1}{2}(e'_1 + e'_n)^2(P_{e_1}(e_n)) \\
&= \frac{1}{2}(e'_1 + e'_n)(e_1).(e'_1 + e'_n)(e_n) = \frac{1}{2}
\end{aligned}$$

Thus the linear operator P_{e_1} is not completely continuous and therefore it is not absolutely 1-summing (see [DJT95, Theorem 2.17]).

The same example shows, for real Banach spaces, the following (see [Dim03, Example 3.4]):

Example 2.1.15. *Strongly 1-summing polynomials are not compatible with the ideal of absolutely 1-summing operators.*

2.1.1 Existence of a compatible operator ideal

We know that there can be more than one ideal of polynomials compatible with a given polynomial ideal, for example the ideals of 2-dominated and multiple 2-summing 2-homogeneous polynomials are both compatible with the ideal of absolutely 2-summing operators (we will see in the next section that this is true for every operator ideal). Also, by Proposition 2.1.5 there exist at most one operator ideal compatible with a given polynomial ideal. On the other hand not every polynomial ideal is compatible with the commonly associated operator ideal (e.g. the absolutely 1-summing polynomials above).

So it is natural to ask wether every polynomial ideal must have a (necessarily unique) compatible operator ideal or not. We will now answer this question affirmatively, proving the following:

Theorem 2.1.16. *Let $\mathfrak{A}_n$ be a Banach ideal of n-homogeneous polynomials. Then there exists a unique Banach ideal of operators $\mathfrak{A}$ compatible with $\mathfrak{A}_n$. This operator ideal can be normed to obtain compatibility constants $1 \leq A, B \leq e$.*

The proof will be given in several steps. First, we need the following Lemma, which is a variation of Lemma 2.1.1.

Lemma 2.1.17. *Let $\mathfrak{A}_n$ a normed ideal of n-homogeneous polynomials and $P \in \mathfrak{A}_n(E,F)$. If $T_1, \ldots, T_{n-1}, S \in \mathcal{L}(G,E)$, then the n-homogeneous polynomial $Q(\cdot) = \overset{\vee}{P}(T_1(\cdot), \cdots, T_{n-1}(\cdot), S(\cdot))$ belongs to $\mathfrak{A}_n(G,F)$. If $T_1 = \cdots = T_{n-1}$ then $\|Q\|_{\mathfrak{A}_n(G,F)} \leq e\|T\|^{n-1}_{\mathcal{L}(G,E)}\|S\|_{\mathcal{L}(G,E)}\|P\|_{\mathfrak{A}_n(E,F)}$.*

In particular, if $S \in \mathcal{L}(G,E)$, $\gamma_1, \ldots, \gamma_k \in E'$, $k < n$ and $a \in E$, then $\gamma_1 \ldots \gamma_k(P_{a^k} \circ S) \in \mathfrak{A}_n(G,F)$; and if $\gamma \in E'$ then:

(a) $\gamma^{n-1}(P_{a^{n-1}} \circ S) \in \mathfrak{A}_n(G,F)$ *with* $\|\gamma^{n-1}(P_{a^{n-1}} \circ S)\|_{\mathfrak{A}_n(G,F)} \leq e\|\gamma\|^{n-1}\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)}\|S\|_{\mathcal{L}(G,E)}$.

(b) $\gamma(P_a \circ S) \in \mathfrak{A}_n(E,F)$ *with* $\|\gamma(P_a \circ S)\|_{\mathfrak{A}_n(E,F)} \leq e\|\gamma\|\|a\|\|P\|_{\mathfrak{A}_n(E,F)}\|S\|^{n-1}$.

Proof. It remains to prove the norm inequality in the case $T_1 = \cdots = T_{n-1} = T$, since the rest was proved in Lemma 2.1.1. Suppose moreover that $\|S\| = \|T\| = 1$.

As in Corollary 2.1.7, we can write Q in the following useful way:

$$Q(x) = \frac{1}{n^2}\frac{1}{(n-1)^{n-1}}\sum_{j=0}^{n-1} r^j P\left((n-1)r^j T(x) + S(x)\right),$$

where r is a primary root of the unit. Thus, defining, for each $0 \leq j \leq n-1$, the linear operator

$$S_j(x) = (n-1)r^j T(x) + S(x),$$

we have that

$$Q = \frac{1}{n^2}\frac{1}{(n-1)^{n-1}}\sum_{j=0}^{n-1} r^j \big(P \circ S_j\big).$$

Therefore, Q belongs to $\mathfrak{A}_n(G,F)$.

For the estimation of the norm, it is enough to consider the case $\|S\| = \|T\| = 1$. Since $\|S_j\| \le n$, for every $j = 0, \ldots, n-1$, we obtain

$$\|Q\|_{\mathfrak{A}_n(G,F)} \le \frac{1}{n^2}\frac{1}{(n-1)^{n-1}} n\|P\|_{\mathfrak{A}_n(G,F)} n^n = \frac{n^{n-1}}{(n-1)^{n-1}}\|P\|_{\mathfrak{A}_n(G,F)} \le e\|P\|_{\mathfrak{A}_n(G,F)}.$$

For the particular cases, just note that $\gamma^{n-1}\left(P_{a^{n-1}} \circ S\right)(x) = \overset{\vee}{P}(\gamma(x)a, \cdots, \gamma(x)a, S(x))$, and $\gamma(P_a \circ S)(x) = \overset{\vee}{P}(\gamma(x)a, S(x), \cdots, S(x))$. $\square$

As a consequence of this lemma we obtain the following.

Lemma 2.1.18. *Let $\mathfrak{A}_n$ be an ideal of n-homogeneous polynomials, let $T \in \mathcal{L}(E,F)$ and fix a nonzero $\gamma_0 \in E'$. Then $\gamma_0^{n-1}T \in \mathfrak{A}_n(E,F)$ if and only if $\gamma^{n-1}T \in \mathfrak{A}_n(E,F)$ for every $\gamma \in E'$.*

Proof. Pick $a \in E$ such that $\gamma_0(a) \neq 0$. By Lemma 2.1.17, $\gamma^{n-1}\left(\gamma_0^{n-1}T\right)_{a^{n-1}} \in \mathfrak{A}_n(E,F)$. We have

$$\gamma^{n-1}\left(\gamma_0^{n-1}T\right)_{a^{n-1}}(x) = \frac{\gamma(x)^{n-1}}{n}\Big(\gamma_0(a)^{n-1}T(x) + (n-1)\gamma_0(a)^{n-2}\gamma_0(x)T(a)\Big).$$

Therefore

$$\left(\gamma^{n-1}T\right)(\cdot) = \frac{n}{\gamma_0(a)^{n-1}}\left(\gamma^{n-1}(\cdot)(\gamma_0^{n-1}T)_{a^{n-1}}(\cdot) - \frac{n-1}{n}\gamma^{n-1}(\cdot)\gamma_0(\cdot)\gamma_0(a)^{n-2}T(a)\right),$$

and then $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E,F)$. $\square$

Now we can define, for a fixed polynomial ideal $\mathfrak{A}_n$, an operator ideal $\mathfrak{A}$, and a complete norm on it. This norm also has some interesting properties that we present in the following proposition.

Proposition 2.1.19. *Let $\mathfrak{A}_n$ be an ideal of n-homogeneous polynomials. Define, for each pair of Banach spaces E and F,*

$$\mathfrak{A}(E,F) = \left\{T \in \mathcal{L}(E,F) /\ \gamma^{n-1}T \in \mathfrak{A}_n(E,F) \text{ for all } \gamma \in E'\right\},$$

with $|||T|||_{\mathfrak{A}(E,F)} = \sup_{\gamma \in S_{E'}} \|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)}$. Then

(a) $\mathfrak{A}$ is an ideal of operators and $\mathfrak{A}(E,F) = \{P_{a^{n-1}} \in \mathcal{L}(E,F) /\ P \in \mathfrak{A}_n(E,F), a \in E\}$.

(b) $|||\cdot|||_{\mathfrak{A}(E,F)}$ is a norm on $\mathfrak{A}(E,F)$ and satisfies

$$|||T|||_{\mathfrak{A}(E,F)} \ge \|T\|_{\mathcal{L}(E,F)}, \quad \text{for every } T \in \mathfrak{A}(E,F).$$

Moreover, $\Big(\mathfrak{A}(E,F), |||\cdot|||_{\mathfrak{A}(E,F)}\Big)$ is a Banach space.

(c) $|||S \circ T|||_{\mathfrak{A}(E,F_1)} \le \|S\|_{\mathcal{L}(F,F_1)}|||T|||_{\mathfrak{A}(E,F)}$ for every $S \in \mathcal{L}(F,F_1)$ and $T \in \mathfrak{A}(E,F)$.

(d) If E_0 is a subspace of E with norm 1 inclusion $i : E_0 \hookrightarrow E$, then

$$|||T \circ i|||_{\mathfrak{A}(E_0,F)} \le |||T|||_{\mathfrak{A}(E,F)}, \quad \text{for all } T \in \mathfrak{A}(E,F).$$

Proof. (a) Clearly the sum and multiplication by scalars of members of $\mathfrak{A}$ is again in $\mathfrak{A}$. So, to prove that $\mathfrak{A}$ is an ideal of operators, we have to show that it behaves well with compositions.

Consider $T \in \mathfrak{A}(E,F)$, $R \in \mathcal{L}(E_1,E)$ and $S \in \mathcal{L}(F,F_1)$. Let us prove that $S \circ T \circ R \in \mathfrak{A}(E_1,F_1)$. Let $\gamma \in E'$ such that $\gamma \circ R \neq 0$. Then $\gamma^{n-1}T \in \mathfrak{A}_n(E,F)$ and $\eta = \gamma \circ R \in E_1'$. By Lemma 2.1.18, it suffices to show that $\eta^{n-1}\big(S \circ T \circ R\big) \in \mathfrak{A}_n(E_1,F_1)$. This follows from the equalities:

$$\Big(\eta^{n-1}\big(S \circ T \circ R\big)\Big)(x) = \gamma^{n-1}\big(R(x)\big)S\big(T\big(R(x)\big)\big) = \big(S \circ (\gamma^{n-1}T) \circ R\big)(x).$$

Therefore $\mathfrak{A}$ is an ideal of operators.

To prove the equivalent definition of $\mathfrak{A}$, suppose $T = P_{a^{n-1}}$ with $P \in \mathfrak{A}_n(E,F)$ and $a \in E$. Then by Lemma 2.1.17, $\gamma^{n-1}T$ belongs to $\mathfrak{A}_n(E,F)$, for all $\gamma \in E'$, and thus $T \in \mathfrak{A}(E,F)$.

Conversely, if $T \in \mathfrak{A}(E,F)$ then $\gamma^{n-1}T \in \mathfrak{A}_n(E,F)$ for every $\gamma \in E'$. Let $a \in E$ such that $\gamma(a) = 1$, then $P = n\gamma^{n-1}T - (n-1)T(a)\gamma^n$ is in $\mathfrak{A}_n(E,F)$ and $P_{a^{n-1}} = T$.

(b) It is straightforward to prove that we defined a norm.

Let $T \in \mathfrak{A}(E,F)$, take $x \in S_E$ such that $\|T(x)\| > \|T\|_{\mathcal{L}(E,F)} - \varepsilon$ and $\gamma \in S_{E'}$ such that $|\gamma(x)| = 1$. Then,

$$|||T|||_{\mathfrak{A}(E,F)} \geq \|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)} \geq \|\gamma^{n-1}T\|_{\mathcal{P}^n(E,F)} \geq \|\gamma(x)^{n-1}T(x)\| > \|T\|_{\mathcal{L}(E,F)} - \varepsilon.$$

Since this is true for every $\varepsilon > 0$, we have that $|||T|||_{\mathfrak{A}(E,F)} \geq \|T\|_{\mathcal{L}(E,F)}$.

Let us see that $\Big(\mathfrak{A}(E,F), |||\cdot|||_{\mathfrak{A}(E,F)}\Big)$ is complete. Suppose $\sum_{k\in\mathbb{N}} |||T_k|||_{\mathfrak{A}(E,F)}$ is convergent. Then $\sum_{k\in\mathbb{N}} \|T_k\|_{\mathcal{L}(E,F)}$ is convergent. Therefore there exists $T \in \mathcal{L}(E,F)$ such that $\sum_k T_k \to T$ in $\mathcal{L}(E,F)$.

For each $\gamma \in S_{E'}$, we know that $\gamma^{n-1}T_k \in \mathfrak{A}_n(E,F)$ and $\|\gamma^{n-1}T_k\|_{\mathfrak{A}_n(E,F)} \leq |||T_k|||_{\mathfrak{A}(E,F)}$. Thus, $\sum_k \gamma^{n-1}T_k$ converges in $\mathfrak{A}_n(E,F)$ and its limit has to be $\gamma^{n-1}T$. Therefore, T belongs to $\mathfrak{A}(E,F)$. Moreover, since

$$\sup_{\gamma\in S_{E'}} \left\|\gamma^{n-1}\sum_{k\geq N} T_k\right\|_{\mathfrak{A}_n(E,F)} \leq \sup_{\gamma\in S_{E'}} \sum_{k\geq N} \left\|\gamma^{n-1}T_k\right\|_{\mathfrak{A}_n(E,F)} \leq \sum_{k\geq N} |||T_k|||_{\mathfrak{A}(E,F)} \to 0,$$

as $N \to \infty$, we have that $\sum_k T_k \to T$ in $\Big(\mathfrak{A}(E,F), |||\cdot|||_{\mathfrak{A}(E,F)}\Big)$.

(c) For every $S \in \mathcal{L}(F,F_1)$ and $T \in \mathfrak{A}(E,F)$, we have:

$$\begin{aligned} |||S \circ T|||_{\mathfrak{A}(E,F_1)} &= \sup_{\gamma\in S_{E'}} \|\gamma^{n-1}S \circ T\|_{\mathfrak{A}_n(E,F_1)} = \sup_{\gamma\in S_{E'}} \|S \circ (\gamma^{n-1}T)\|_{\mathfrak{A}_n(E,F_1)} \\ &\leq \|S\|_{\mathcal{L}(F,F_1)} \sup_{\gamma\in S_{E'}} \|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)} = \|S\|_{\mathcal{L}(F,F_1)} |||T|||_{\mathfrak{A}(E,F)}. \end{aligned}$$

(d) Let $T \in \mathfrak{A}(E,F)$ and $\gamma \in E_0'$. Consider $\tilde{\gamma} \in E'$ a Hahn-Banach extension of γ preserving its norm. Then

$$\left\|\gamma^{n-1}(T \circ i)\right\|_{\mathfrak{A}_n(E_0,F)} = \left\|(\tilde{\gamma} \circ i)^{n-1}(T \circ i)\right\|_{\mathfrak{A}_n(E_0,F)} = \left\|(\tilde{\gamma}^{n-1}T) \circ i\right\|_{\mathfrak{A}_n(E_0,F)} \leq \|\tilde{\gamma}^{n-1}T\|_{\mathfrak{A}_n(E,F)}.$$

Taking supremum we have that

$$|||T \circ i|||_{\mathfrak{A}(E_0,F)} \leq |||T|||_{\mathfrak{A}(E,F)}.$$

$\square$

The following proposition shows that the norm defined on $\mathfrak{A}$ is "almost ideal", in the sense that satisfies the ideal condition up to a constant.

Proposition 2.1.20. *The norm $|||\cdot|||_{\mathfrak{A}}$ defined on Proposition 2.1.19 satisfies the "almost ideal" property: for Banach spaces E and F, there exists a constant $c > 0$ such that, for all Banach spaces E_1, F_1 and all operators $R \in \mathcal{L}(E_1, E)$, $T \in \mathfrak{A}(E, F)$ and $S \in \mathcal{L}(F, F_1)$, it follows that*

$$|||S \circ T \circ R|||_{\mathfrak{A}(E_1,F_1)} \leq c\|S\|_{\mathcal{L}(F,F_1)}|||T|||_{\mathfrak{A}(E,F)}\|R\|_{\mathcal{L}(E_1,E)}.$$

Proof. The left composition was proved in Proposition 2.1.19 (c).

For a fixed Banach space E_1 and a fixed operator $R \in \mathcal{L}(E_1, E)$, consider

$$\begin{array}{ccc} \big(\mathfrak{A}(E,F), |||\cdot|||_{\mathfrak{A}(E,F)}\big) & \to & \big(\mathfrak{A}(E_1,F), |||\cdot|||_{\mathfrak{A}(E_1,F)}\big) \\ T & \mapsto & T \circ R \end{array}$$

An application of the Closed Graph Theorem gives the existence of a constant $c_{E_1,R} > 0$ such that

$$|||T \circ R|||_{\mathfrak{A}(E_1,F)} \leq c_{E_1,R}|||T|||_{\mathfrak{A}(E,F)}.$$

If we apply again the Closed Graph Theorem for

$$\begin{array}{ccc} \mathcal{L}(E_1,E) & \to & \mathcal{L}(\mathfrak{A}(E,F), \mathfrak{A}(E_1,F)) \\ R & \mapsto & \theta_R(T) = T \circ R, \end{array}$$

we obtain that there is a constant $c_{E_1} > 0$ such that

$$|||T \circ R|||_{\mathfrak{A}(E_1,F)} \leq c_{E_1}|||T|||_{\mathfrak{A}(E,F)}\|R\|_{\mathcal{L}(E_1,E)}. \tag{2.3}$$

Now suppose that the result is not true. Then there are Banach spaces E_k, and $R_k \in \mathcal{L}(E_k, E)$, $\|R_k\|_{\mathcal{L}(E_k,E)} = 1$, for all $k \in \mathbb{N}$, such that

$$|||T \circ R_k|||_{\mathfrak{A}(E_k,F)} > k.$$

Let $E_0 = \bigoplus_{k\in\mathbb{N}} E_k$, and $\tilde{R}_k \in \mathcal{L}(E_0, E)$, $\tilde{R}_k = R_k \circ \pi_k$, where $\pi_k : E_0 \to E_k$ is the (norm one) projection. Denote by $i_k : E_k \hookrightarrow E_0$ the (norm one) inclusion. So we have

$$\begin{aligned} k &< |||T \circ R_k|||_{\mathfrak{A}(E_k,F)} = |||T \circ R_k \circ \pi_k \circ i_k|||_{\mathfrak{A}(E_k,F)} \\ &= |||T \circ \tilde{R}_k \circ i_k|||_{\mathfrak{A}(E_k,F)} \leq |||T \circ \tilde{R}_k|||_{\mathfrak{A}(E_0,F)}, \end{aligned}$$

the last inequality following from Proposition 2.1.19(d). Also, by (2.3),

$$|||T \circ \tilde{R}_k|||_{\mathfrak{A}(E_0,F)} \leq c_{E_0}|||T|||_{\mathfrak{A}(E,F)}\|\tilde{R}_k\|_{\mathcal{L}(E_0,E)} \leq c_{E_0}|||T|||_{\mathfrak{A}(E,F)},$$

which leads to a contradiction.

□

Now we present a result that shows how to convert an "almost ideal" norm into an ideal norm.

Proposition 2.1.21. *Let $\mathfrak{A}$ be an operator ideal with norm $|||\cdot|||_{\mathfrak{A}}$ that satisfies the "almost ideal" property. Then we can define an equivalent norm $\|\cdot\|_{\mathfrak{A}}$ which is an ideal norm on $\mathfrak{A}$.*

Proof. We first define a norm $\|\cdot\|'_{\mathfrak{A}}$ in the following way. For $T \in \mathfrak{A}(E,F)$, let

$$\|T\|'_{\mathfrak{A}(E,F)} = \sup\{|||S \circ T \circ R|||_{\mathfrak{A}(E_1,F_1)} : \ E_1, F_1 \text{ Banach spaces}, \|S\|_{\mathcal{L}(F,F_1)} = \|R\|_{\mathcal{L}(E_1,E)} = 1\}.$$

It is easy to see that $\|\cdot\|'_{\mathfrak{A}}$ is a norm on $\mathfrak{A}$ equivalent to $|||\cdot|||_{\mathfrak{A}}$. Also, it is clear that satisfies the ideal property:

$$\|S \circ T \circ R\|'_{\mathfrak{A}(E_1,F_1)} \leq \|S\|_{\mathcal{L}(F,F_1)} \|T\|'_{\mathfrak{A}(E,F)} \|R\|_{\mathcal{L}(E_1,E)}.$$

Last, if $\kappa = \|id_{\mathbb{C}}\|'_{\mathfrak{A}(\mathbb{C},\mathbb{C})}$ then the norm $\|\cdot\|_{\mathfrak{A}}$ defined by

$$\|T\|_{\mathfrak{A}(E,F)} = \frac{1}{\kappa}\|T\|'_{\mathfrak{A}(E,F)}$$

is an ideal norm equivalent to $|||\cdot|||_{\mathfrak{A}}$. □

Remark 2.1.22. When applying the previous proposition to our context (that is, $\mathfrak{A}_n$ a polynomial ideal and $(\mathfrak{A}, |||\cdot|||_{\mathfrak{A}})$ as in Proposition 2.1.19), using Proposition 2.1.19 (*ii*), we can simplify the definition of $\|\cdot\|'_{\mathfrak{A}}$:

$$\|T\|'_{\mathfrak{A}(E,F)} = \sup\{|||T \circ R|||_{\mathfrak{A}(E_1,F)} : \ E_1 \text{ Banach space}, \|R\|_{\mathcal{L}(E_1,E)} = 1\}.$$

Then considering

$$\|T\|_{\mathfrak{A}(E,F)} = \frac{\|T\|'_{\mathfrak{A}(E,F)}}{\|id_{\mathbb{C}}\|'_{\mathfrak{A}(\mathbb{C},\mathbb{C})}}$$

we obtain an ideal norm on $\mathfrak{A}$ equivalent to $|||\cdot|||_{\mathfrak{A}}$. Moreover,

$$\begin{aligned} \kappa &= \|z \mapsto z\|'_{\mathfrak{A}(\mathbb{C},\mathbb{C})} = \sup\{|||(z \mapsto z) \circ \varphi|||_{\mathfrak{A}(E_1,\mathbb{C})} : \ E_1 \text{ Banach space}, \varphi \in S_{E_1'}\} \\ &= \sup\{|||\varphi|||_{\mathfrak{A}(E_1,\mathbb{C})} : \ E_1 \text{ Banach space}, \varphi \in S_{E_1'}\} \\ &= \sup\{\|\gamma^{n-1}\varphi\|_{\mathfrak{A}_n(E_1,\mathbb{C})} : \ E_1 \text{ Banach space}, \varphi, \gamma \in S_{E_1'}\}. \end{aligned}$$

Thus by Corollary 2.1.7 we have that $1 \leq \kappa \leq e$. □

We now can prove the existence, for any polynomial ideal, of a compatible operator ideal:

Proof. (of Theorem 2.1.16) Consider the normed ideal $(\mathfrak{A}, \|\cdot\|_{\mathfrak{A}})$, with

$$\mathfrak{A}(E,F) = \left\{T \in \mathcal{L}(E,F) / \ \gamma^{n-1}T \in \mathfrak{A}_n(E,F) \text{ for all } \gamma \in E'\right\}$$

and $\|\cdot\|_{\mathfrak{A}}$ given by Remark 2.1.22 (*ii*). By the equivalence with $|||\cdot|||_{\mathfrak{A}}$ and Proposition 2.1.19 (*b*), for each E and F Banach, $\big(\mathfrak{A}(E,F), \|\cdot\|_{\mathfrak{A}(E,F)}\big)$ is a Banach space.

Let us check that $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$.

It is clear, by definition, that if $T \in \mathfrak{A}(E,F)$ and $\gamma \in E'$ then $\gamma^{n-1}T \in \mathfrak{A}_n(E,F)$. On the other hand take $P \in \mathfrak{A}_n(E,F)$ and $a \in E$. By Proposition 2.1.19 (*a*), $P_{a^{n-1}}$ belongs to $\mathfrak{A}(E,F)$. By Proposition 2.1.3 we conclude that $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$. We can moreover estimate the constants of compatibility. For the first one, by Lemma 2.1.17 (*a*),

$$\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} = \frac{1}{\kappa} \sup_{\substack{E_1 \ Banach \\ R \in S_{\mathcal{L}(E_1,E)}}} \sup_{\|\gamma\|=1} \left\|\gamma^{n-1}(P_{a^{n-1}} \circ R)\right\|_{\mathfrak{A}_n(E_1,F)} \leq \frac{e}{\kappa}\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)}.$$

For the other constant we have,

$$\|\gamma^{n-1}T\|_{\mathfrak{A}_n(E,F)} = \|\gamma\|^{n-1}\left\|\frac{\gamma^{n-1}}{\|\gamma\|^{n-1}}T\right\|_{\mathfrak{A}_n(E,F)} \leq \|\gamma\|^{n-1}|||T|||_{\mathfrak{A}(E,F)} \leq \kappa\|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E,F)}.$$

The fact that $\mathfrak{A}$ is the only ideal of operators compatible with $\mathfrak{A}_n$ follows from Proposition 2.1.5. □

Thus every polynomial Banach ideal is compatible with an operator ideal. We showed that absolutely 1-summing polynomials are not compatible with absolutely 1-summing operators. Then the question that comes up now is which is the ideal of linear operators which is compatible with the absolutely 1-summing polynomials.

As the following example shows, the unique compatible operator ideal may be far from "natural". Note, however, that this unnatural compatibility has some interesting consequences.

Example 2.1.23. *The ideal Π_p^n of absolutely-p-summing n-homogeneous polynomials is compatible with $\mathcal{L}$, the ideal of continuous linear operators, with constants $A = e$ and $B = 1$.*

Proof. Obviously, for $P \in \Pi_p^n(E, F)$ and $a \in E$, $P_{a^{n-1}}$ belongs to $\mathcal{L}(E, F)$ and

$$\|P_{a^{n-1}}\|_{\mathcal{L}(E,F)} \leq e\|P\|_{\mathcal{P}^n(E,F)}\|a\|^{n-1} \leq e\|P\|_{\Pi_p^n(E,F)}\|a\|^{n-1}.$$

For the other condition, let $T \in \mathcal{L}(E, F)$ and $\gamma \in E'$, then, for all $x_1, \ldots, x_m \in E$,

$$\begin{aligned}\Big(\sum_{j=1}^m \|(\gamma^{n-1}T)(x_j)\|^p\Big)^{\frac{1}{p}} &\leq \|\gamma\| \left(\sum_{j=1}^m \Big(\frac{|\gamma(x_j)|}{\|\gamma\|}\|\gamma\|^{n-2}\|T\|\|x_j\|^{n-1}\Big)^p\right)^{\frac{1}{p}} \\ &\leq \|\gamma\|^{n-1}\|T\| \left(\sum_{j=1}^m \Big(\frac{|\gamma(x_j)|}{\|\gamma\|}\Big)^p\right)^{\frac{1}{p}} \Big(\max_{1\leq j\leq m}\|x_j\|\Big)^{n-1} \\ &\leq \|\gamma\|^{n-1}\|T\| \sup_{x'\in B_{E'}} \left(\sum_{j=1}^m |x'(x_j)|^p\right)^{\frac{n}{p}} = \|\gamma\|^{n-1}\|T\|\omega_p\big((x_j)_{j=1}^m\big)^n.\end{aligned}$$

Thus, $\gamma^{n-1}T$ is absolutely p-summing and

$$\|\gamma^{n-1}T\|_{\Pi_p^n(E,F)} \leq \|T\|_{\mathcal{L}(E,F)}\|\gamma\|^{n-1}.$$

□

Corollary 2.1.24. *Suppose that $\Pi_p^n(E, F) \subset \mathfrak{A}_n(E, F)$ and that $\mathfrak{A}_n$ is compatible with $\mathfrak{A}_1$. Then $\mathfrak{A}_1(E, F) = \mathcal{L}(E, F)$.*

Proof. This is just a special case of Proposition 2.1.5. □

It is well known that every absolutely summing operator is weakly compact (see for example [DJT95, Theorem 2.17]). In [Bot02] it was shown that not every dominated polynomial is weakly compact by exhibiting an example of a polynomial from ℓ_1 to ℓ_1. We now show how the concept of compatible ideals can be easily applied to prove that not every absolutely p-summing homogeneous polynomial is weakly compact.

Corollary 2.1.25. *E is reflexive if and only if every absolutely p-summing n-homogeneous polynomial (from E to E, $n \geq 2$) is weakly compact.*

Proof. We know from the examples that the weakly compact homogeneous polynomials are compatible with the weakly compact operators ($\mathcal{L}_{WK} \sim \mathcal{P}_{WK}^n$). Suppose that $\Pi_p^n(E, E) \subset \mathcal{P}_{WK}^n(E, E)$. Then, by the previous Corollary, we have that $\mathcal{L}(E, E) = \mathcal{L}_{WK}(E, E)$ and thus E must be reflexive.

Conversely, if E is reflexive, every homogeneous polynomial is weakly compact. □

Analogously we can prove that every absolutely p-summing n-homogeneous polynomial from E to F ($n \geq 2$) is weakly compact if and only if every linear operator from E to F is weakly compact.

2.2 The smallest and the largest compatible ideals

Consider a normed ideal $\mathfrak{A}$ of linear operators. In this section we define normed ideals of n-homogeneous polynomials, $\mathcal{M}_n^{\mathfrak{A}}$ and $\mathcal{F}_n^{\mathfrak{A}}$, compatible with $\mathfrak{A}$ with the following property: if $\mathfrak{A}_n$ is another ideal compatible with $\mathfrak{A}$ then for each E, F,

$$\mathcal{F}_n^{\mathfrak{A}}(E,F) \subset \mathfrak{A}_n(E,F) \subset \mathcal{M}_n^{\mathfrak{A}}(E,F).$$

In other words, $\mathcal{M}_n^{\mathfrak{A}}$ y $\mathcal{F}_n^{\mathfrak{A}}$ are, respectively, the largest and the smallest ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$.

Define, for Banach spaces E and F,

$$\mathcal{M}_n^{\mathfrak{A}}(E,F) = \big\{P \in \mathcal{P}^n(E,F) \ / \ P_{a^{n-1}} \in \mathfrak{A}(E,F), \ \forall\, a \in E\big\} \tag{2.4}$$

with norm

$$\|P\|_{\mathcal{M}_n^{\mathfrak{A}}(E,F)} := \sup_{\|a\|=1} \|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)}.$$

Also, we define

$$\mathcal{F}_n^{\mathfrak{A}}(E,F) = \big\{P \in \mathcal{P}^n(E,F) \ / \ P = \sum_{i=1}^{m} \gamma_i^{n-1} T_i, \ T_i \in \mathfrak{A}(E,F), \ \gamma_i \in E'\big\} \tag{2.5}$$

with norm

$$\|P\|_{\mathcal{F}_n^{\mathfrak{A}}(E,F)} := \inf \left\{ \sum_{i=1}^{m} \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} \right\},$$

where the infimum is taken over all possible representations of P as in equation (2.5).

In the case of $\mathfrak{A}$ being complete, we define also

$$\mathcal{N}_n^{\mathfrak{A}}(E,F) = \big\{P \in \mathcal{P}^n(E,F) \ / \ P = \sum_{i=1}^{\infty} \gamma_i^{n-1} T_i\big\} \tag{2.6}$$

where $T_i \in \mathfrak{A}(E,F)$, $\gamma_i \in E'$ and $\sum_{i=1}^{\infty} \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} < \infty$, with norm

$$\|P\|_{\mathcal{N}_n^{\mathfrak{A}}(E,F)} := \inf \left\{ \sum_{i=1}^{\infty} \|\gamma_i\|^{n-1} \|T_i\|_{\mathfrak{A}(E,F)} \right\},$$

where the infimum is taken over all possible representations of P as in equation (2.6).

Remark 2.2.1. It is easy to prove the following isometric identifications for the previously defined ideals:

$$\mathcal{M}_n^{\mathfrak{A}}(E,F) \overset{1}{=} \mathcal{P}^{n-1}(E, \mathfrak{A}(E,F)),$$

$$\mathcal{F}_n^{\mathfrak{A}}(E,F) \overset{1}{=} \mathcal{P}_f^{n-1}(E, \mathfrak{A}(E,F)),$$

$$\mathcal{N}_n^{\mathfrak{A}}(E,F) \overset{1}{=} \mathcal{P}_N^{n-1}(E, \mathfrak{A}(E,F)).$$

Now we show that these polynomial ideals are the extreme cases among those compatible with $\mathfrak{A}$:

Proposition 2.2.2. *Let $\mathfrak{A}$ be a normed ideal of linear operators. Then:*

a) $\mathcal{M}_n^{\mathfrak{A}}$ is the largest normed ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$.

b) $\mathcal{F}_n^{\mathfrak{A}}$ is the smallest normed ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$.

c) If $\mathfrak{A}$ is complete, then $\mathcal{N}_n^{\mathfrak{A}}$ is the smallest Banach ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$.

Moreover, in all the cases compatibility constants are $A = B = 1$.

Proof. *a*) It is clear that $\mathcal{M}_n^{\mathfrak{A}}$ is a normed ideal of n-homogeneous polynomials. Moreover, if $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$ and $P \in \mathfrak{A}_n(E,F)$, then $P_{a^{n-1}} \in \mathfrak{A}(E,F)$. Therefore $P \in \mathcal{M}_n^{\mathfrak{A}}(E,F)$ and hence $\mathfrak{A}_n \subset \mathcal{M}_n^{\mathfrak{A}}$.

It remains to verify that $\mathcal{M}_n^{\mathfrak{A}}$ is compatible with $\mathfrak{A}$. Condition (i) is clearly satisfied with constant $A = 1$.

To see that $\mathcal{M}_n^{\mathfrak{A}}$ satisfies (ii), let $T \in \mathfrak{A}(E,F)$, $\gamma \in E'$ and $a \in E$. We have

$$(\gamma^{n-1}T)_{a^{n-1}} = \frac{1}{n}\gamma(a)^{n-1}T + \frac{n-1}{n}\gamma(a)^{n-2}T(a)\gamma.$$

Then, $(\gamma^{n-1}T)_{a^{n-1}} \in \mathfrak{A}(E,F)$ and therefore, $\gamma^{n-1}T \in \mathcal{M}_n^{\mathfrak{A}}(E,F)$. Moreover, by the triangle inequality,

$$\|(\gamma^{n-1}T)_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq \|\gamma\|^{n-1}\|T\|_{\mathfrak{A}(E,F)}\|a\|^{n-1}.$$

Thus (ii) is satisfied with constant $B = 1$.

The proof of *b*) is a simpler version of the proof of *c*).

c) It is easy to see that $\mathcal{N}_n^{\mathfrak{A}}$ is a normed ideal of n-homogeneous polynomials. Completeness follows from Remark 2.2.1.

We now prove that if $\mathfrak{A}_n$ is a Banach ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$ (with constants $\widetilde{A}$ and $\widetilde{B}$) then $\mathcal{N}_n^{\mathfrak{A}}(E,F) \subset \mathfrak{A}_n(E,F)$. Consider $P \in \mathcal{N}_n^{\mathfrak{A}}(E,F)$ with representation $P = \sum_{i=1}^{\infty} \gamma_i^{n-1}T_i$, where $T_i \in \mathfrak{A}(E,F)$, $\gamma_i \in E'$. For every $k \in \mathbb{N}$, by the compatibility of $\mathfrak{A}_n$ with $\mathfrak{A}$, we have that $\sum_{i=1}^{k} \gamma_i^{n-1}T_i \in \mathfrak{A}_n(E,F)$. Moreover, the series is convergent in $\mathfrak{A}_n(E,F)$ since

$$\sum_{i=1}^{\infty} \|\gamma_i^{n-1}T_i\|_{\mathfrak{A}_n(E,F)} \leq \widetilde{B}\sum_{i=1}^{\infty} \|\gamma_i\|^{n-1}\|T_i\|_{\mathfrak{A}(E,F)}.$$

Hence $P \in \mathfrak{A}_n(E,F)$ and $\|P\|_{\mathfrak{A}_n(E,F)} \leq \widetilde{B}\|P\|_{\mathcal{N}_n^{\mathfrak{A}}(E,F)}$.

Finally, we prove that $\mathcal{N}_n^{\mathfrak{A}}$ is compatible with $\mathfrak{A}$. It is immediate that (ii) is satisfied with constant $B = 1$. To prove (i), consider $a \in E$, and $P \in \mathcal{N}_n^{\mathfrak{A}}(E,F)$ and choose a representation $P = \sum_{i=1}^{\infty} \gamma_i^{n-1}T_i$. Then

$$P_{a^{n-1}} = \sum_{i=1}^{\infty}\left[\frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i\right].$$

For each $k \in \mathbb{N}$, $\sum_{i=1}^{k}\left[\frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i\right] \in \mathfrak{A}(E,F)$, and

$$\sum_{i=1}^{\infty}\left\|\frac{1}{n}\gamma_i(a)^{n-1}T_i + \frac{n-1}{n}\gamma_i(a)^{n-2}T_i(a)\gamma_i\right\|_{\mathfrak{A}(E,F)} \leq \left(\sum_{i=1}^{\infty}\|\gamma_i\|^{n-1}\|T_i\|_{\mathfrak{A}(E,F)}\right)\|a\|^{n-1},$$

for every representation of P. Since $\mathfrak{A}$ is a complete ideal, we obtain that $P_{a^{n-1}} \in \mathfrak{A}(E,F)$ and

$$\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq \|a\|^{n-1}\|P\|_{\mathcal{N}_n^{\mathfrak{A}}(E,F)},$$

for every $a \in E$. Thus (i) is satisfied with constant $A = 1$. $\square$

Remark 2.2.3. Note that if $\mathfrak{A}$ is any operator ideal, the polynomial ideals $\mathcal{M}_n^{\mathfrak{A}}$, $\mathcal{F}_n^{\mathfrak{A}}$ and $\mathcal{N}_n^{\mathfrak{A}}$ are always different. Indeed, in the scalar valued case we have:

$$\mathcal{M}_n^{\mathfrak{A}}(E) = \mathcal{P}^n(E), \quad \mathcal{F}_n^{\mathfrak{A}}(E) = \mathcal{P}_f^n(E) \quad \text{and} \quad \mathcal{N}_n^{\mathfrak{A}}(E) = \mathcal{P}_N^n(E)$$

with equivalent norms. This means, in particular, that for any operator ideal, there are always several different polynomial ideals compatible with it.

The following proposition is an immediate consequence of the definition of $\mathcal{M}_n^{\mathfrak{A}}$, $\mathcal{F}_n^{\mathfrak{A}}$ and $\mathcal{N}_n^{\mathfrak{A}}$. It provides a kind of converse of Proposition 2.1.5 for the special cases of largest and smallest compatible ideals:

Proposition 2.2.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be normed operator ideals. If for some E and F, $\mathfrak{A}(E,F) \subset \mathfrak{B}(E,F)$, then for all $n \geq 1$, $\mathcal{M}_n^{\mathfrak{A}}(E,F) \subset \mathcal{M}_n^{\mathfrak{B}}(E,F)$ and $\mathcal{F}_n^{\mathfrak{A}}(E,F) \subset \mathcal{F}_n^{\mathfrak{B}}(E,F)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are complete, then we have also that $\mathcal{N}_n^{\mathfrak{A}}(E,F) \subset \mathcal{N}_n^{\mathfrak{B}}(E,F)$.*

Proposition 2.2.2 allows us to obtain the following characterization of all the polynomial ideals which are compatible with a given operator ideal.

Proposition 2.2.5. *Let $\mathfrak{A}$ be a normed operator ideal and $\mathfrak{A}_n$ a normed ideal of n-homogeneous polynomials. Then the following are equivalent*
(i) $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$ with constants A and B
(ii) For every Banach space E, F, we have inclusions

$$\mathcal{F}_n^{\mathfrak{A}}(E,F) \overset{j_1}{\hookrightarrow} \mathfrak{A}_n(E,F) \overset{j_2}{\hookrightarrow} \mathcal{M}_n^{\mathfrak{A}}(E,F),$$

with $\|j_1\| \leq A$ and $\|j_2\| \leq B$.

2.3 Composition ideals

In this section we relate the compatibility concept with composition ideals. We prove that if we have compatible ideals and compose them with closed operator ideals then the resulting ideals are compatible. A similar result follows if we compose any operator ideal with a closed polynomial ideal.

Proposition 2.3.1. *Let $\mathfrak{A}$ be a normed ideal of linear operators and $\mathfrak{A}_n$ a normed ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$ with constants A and B. If $\mathfrak{C}$ and $\mathfrak{B}$ are closed operator ideals, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is compatible with $\mathfrak{C} \circ \mathfrak{A} \circ \mathfrak{B}$ with constants A and B.*

The proof is analogous to the proof of Proposition 3.1.21, so we will not do it here.

Now we turn our attention to the composition of a closed polynomial ideal with arbitrary operator ideals. As a particular case, if $\mathfrak{B}$ is an operator ideal, the composition ideal $\mathcal{P}^n \circ \mathfrak{B}$ is analogous to the Pietsch Factorization method of multilinear mappings. In [BBJP06] the authors norm the polynomial ideal obtained by this method considering in equation (1.1) all the factorizations of the multilinear operator $\check{P}$ rather that the factorizations of P. That is, the norm for the polynomial ideal keeps the multilinear essence of Pietsch method. Although for a fixed n the norm given by (1.1) and the norm in [BBJP06] are equivalent, the equivalence constants depend on n. Therefore, compatibility constants for $\mathcal{P}^n \circ \mathfrak{B}$ do not follow from the analogous properties of the norm defined in [BBJP06]. The following proposition can be proved as Proposition 3.1.22.

Proposition 2.3.2. *Let $\mathfrak{A}_n$ be a closed ideal of n-homogeneous polynomials compatible with a closed operator ideal $\mathfrak{A}$. If $\mathfrak{B}$ and $\mathfrak{C}$ are normed ideals of linear operators, then $\mathfrak{C} \circ \mathfrak{A}_n \circ \mathfrak{B}$ is compatible with $\mathfrak{C} \circ \mathfrak{A} \circ \mathfrak{B}$ with constants $A = e$ and $B = 1$.*

Example 2.3.3. Let $1 < r < \infty$, the ideal of r-dominated n-homogeneous polynomials $\mathcal{D}_r^n$ is the composition ideal $\mathcal{P}^n \circ \Pi_r$ [Sch91], where Π_r is the ideal of absolutely r-summing operators. Thus, by above proposition, $\mathcal{D}_r^n$ is compatible with Π_r, with constants $A = e$ and $B = 1$.

Similarly, for $1 \leq r \leq \infty$, the ideal $\mathcal{P}_r^n$ of r-factorable n-homogeneous polynomials is compatible with the ideal of r-factorable operators with constants $A = e$ and $B = 1$.

2.4 Interpolation of ideals

In this section we will show that interpolation of compatible ideals give as a result new compatible ideals. We first recall some facts about interpolation theory (see [BL76]).

Let X_0 and X_1 be two normed spaces. We say that $\overline{X} = (X_0, X_1)$ is a compatible couple if they are both subspaces of a Hausdorff topological vector space. Then we can form their sum $X_0 + X_1$ and their intersection $X_0 \cap X_1$. If we define

$$\|a\|_{X_0 \cap X_1} = \max\{\|a\|_{X_0} \|a\|_{X_1}\}, \qquad \|a\|_{X_0 + X_1} = \inf_{a = a_0 + a_1} \{\|a_0\|_{X_0} \|a_1\|_{X_1}\},$$

then $X_0 + X_1$ and $X_0 \cap X_1$ become normed spaces, which are complete if X_0 and X_1 are comlete. A normed space X is an intermediate space between X_0 and X_1 (or with respect to $\overline{X}$) if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous inclusions.

Example 2.4.1. For $\mathfrak{A}_n, \mathfrak{B}_n$ any pair of normed ideal of n-homogeneous polynomial, E, F any Banach spaces, the couple $(\mathfrak{A}_n(E, F), \mathfrak{B}_n(E, F))$ is a compatible couple, since they are included in the space of continuous n-homogeneous polynomials $\mathcal{P}^n(E, F)$.

If $\mathfrak{A}_n$ be any normed ideal n-homogeneous polynomial, then $\mathfrak{A}_n(E, F)$ is an intermediate space for the compatible couple $(\mathcal{P}_N^n(E, F), \mathcal{P}^n(E, F))$.

Remark 2.4.2. *Let $\mathfrak{A}$ be a normed ideal of operators, and $\mathfrak{A}_n$ an normed ideal n-homogeneous polynomials, then Proposition 2.2.5 may be rephrased as: $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$ (at (E, F)) if and only if $\mathfrak{A}_n(E, F)$ is an intermediate space for the couple $(\mathcal{F}_n^{\mathfrak{A}}(E, F), \mathcal{M}_n^{\mathfrak{A}}(E, F))$.*

Let $\overline{X}$ and $\overline{Y}$ be two compatible couples, then we say that two spaces X and Y are interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$ if X and Y are intermediate spaces with respect to $\overline{X}$ and $\overline{Y}$ respectively, and if $T : \overline{X} \to \overline{Y}$ (this means that $T|_{X_0} : X_0 \to Y_0$ and $T|_{X_1} : X_1 \to Y_1$ are continuous) implies that $T : X \to Y$ is continuous. If $\|T\|_{X,Y} \leq \|T\|_{X_0,Y_0}^{\theta} \|T\|_{X_1,Y_1}^{1-\theta}$, with $0 \leq \theta \leq 1$ ($\|T\|_{X,Y} \leq \max\{\|T\|_{X_0,Y_0}, \|T\|_{X_1,Y_1}\}$) then X and Y are called exact interpolation spaces of exponent θ (exact interpolation spaces).

An interpolation functor (or interpolation method) is a functor F from the category of compatible couples into the category of normed (Banach) spaces such that if $\overline{X}$ and $\overline{Y}$ are compatible couples then $F(\overline{X})$ and $F(\overline{Y})$ are interpolation spaces with respect to $\overline{X}$ and $\overline{Y}$. Also, $F(T) = T$ for all $T : \overline{X} \to \overline{Y}$. We shall say that F is an exact interpolation functor (of exponent θ) if $F(\overline{X})$ and $F(\overline{Y})$ are exact interpolation spaces (of exponent θ) with respect to $\overline{X}$ and $\overline{Y}$.

The most commonly used functors of interpolation are the complex method and the real methods K and J. We will briefly describe and apply the complex method in Section 3.2.5.

If $\mathfrak{A}_n^0$ and $\mathfrak{A}_n^1$ are normed ideals of n-homogeneous polynomials and F is an exact interpolation functor (or exact of exponent θ) and for each Banach spaces E, F, we denote $F(\mathfrak{A}_n)(E,F) := F(\mathfrak{A}_n(E,F))$ then $F(\mathfrak{A}_n)$ is a normed ideal of n-homogeneous polynomials since the ideal properties can be rephrased as the continuity of certain linear operators. Similarly, the compatibility conditions can be described by the continuity of the linear operators

$$\begin{array}{rl} \mathfrak{A}_n(E,F) \to & \mathfrak{A}(E,F) \\ P & \mapsto P_{a^{n-1}} \end{array} \quad \text{and} \quad \begin{array}{rl} \mathfrak{A}(E,F) \to & \mathfrak{A}_n(E,F) \\ T & \mapsto \gamma^{n-1}T, \end{array}$$

for every $a \in E$, $\gamma \in E'$. Thus we have:

Proposition 2.4.3. *For $i = 0,1$, let $\mathfrak{A}_n^i$ be an ideal of n-homogeneous polynomials compatible with the operator ideal $\mathfrak{A}^i$ with constants A_i and B_i, and let F is an exact interpolation functor (of exponent θ). Then, the polynomial ideal $F(\mathfrak{A}_n)$ is compatible with the operator ideal $F(\mathfrak{A})$ with constants* $\max\{A_0, A_1\}$ *($A_0^{1-\theta}A_1^{\theta}$) and* $\max\{B_0, B_1\}$ *($B_0^{1-\theta}B_1^{\theta}$).*

2.5 Relation with tensor norms

The aim of this section is to relate the concepts of compatibility with tensor norm properties. Given an operator ideal $\mathfrak{A}$, a symmetric n-tensor norm α and Banach spaces E and F, we can define $\mathfrak{A}_n(E,F) \overset{1}{:=} \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$, where we identify P with the linear operator T_P (i.e, $P \in \mathfrak{A}_n(E,F)$ if and only if $T_P \in \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$ and $\|P\|_{\mathfrak{A}_n(E,F)} = \|T_P\|_{\mathfrak{A}(\bigotimes_\alpha^{n,s} E,F)}$). Since α is a tensor norm and $\mathfrak{A}$ an operator ideal, it is easy to see that $\mathfrak{A}_n$ is a normed ideal of n-homogeneous polynomials. We have also

Proposition 2.5.1. *Let α be a symmetric n-tensor norm and $\mathfrak{A}$ an operator ideal. Then, the polynomial ideal $\mathfrak{A}_n(E,F) = \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$ is compatible with $\mathfrak{A}$, with constants $A = e$ and $B = e$.*

Proof. Let us check condition (i) of Definition 2.1.2. Take $P \in \mathfrak{A}_n(E,F)$. For $a \in E$, we define $\Phi_{a^{n-1}} \colon E \to \bigotimes_\alpha^{n,s} E$ by

$$\Phi_{a^{n-1}}(x) = \sigma(a \otimes \cdots \otimes a \otimes x).$$

By Lemma 2.1.6, $\Phi_{a^{n-1}}$ is continuous and

$$\alpha\big(\Phi_{a^{n-1}}(x); \bigotimes^{n,s} E\big) \leq e\|a\|^{n-1}\|x\|.$$

Moreover, $P_{a^{n-1}}(x) = (T_P \circ \Phi_{a^{n-1}})(x)$. Then, $P_{a^{n-1}}$ belongs to $\mathfrak{A}(E,F)$ and

$$\|P_{a^{n-1}}\|_{\mathfrak{A}(E,F)} \leq \|T_P\|_{\mathfrak{A}(E,F)}\|\Phi_{a^{n-1}}\|_{\mathcal{L}(E,\bigotimes_\alpha^{n,s} E)} \leq e\|a\|^{n-1}\|P\|_{\mathfrak{A}_n(E,F)},$$

which gives condition (i) with $A = e$.

Now we prove condition (ii). For $\gamma \in E'$, define $\Psi_{\gamma^{n-1}} : \bigotimes_\alpha^{n,s} E \to E$, as $\Psi_{\gamma^{n-1}}(x^n) = \gamma(x)^{n-1}x$.

To see that $\Psi_{\gamma^{n-1}}$ is continuous, it is enough to consider the case $\alpha = \varepsilon_s$. If $z = \sum_{i=1}^m x_i^n$, we have

$$\begin{aligned} \|\Psi_{\gamma^{n-1}}(z)\|_E &= \left\|\sum_{i=1}^m \gamma(x_i)^{n-1}x_i\right\|_E = \sup_{\varphi \in B_{E'}} \left|\sum_{i=1}^m \gamma(x_i)^{n-1}\varphi(x_i)\right| \\ &= \sup_{\varphi \in B_{E'}} |\langle \gamma^{n-1}\varphi, z\rangle| \leq \sup_{\varphi \in B_{E'}} \|\gamma^{n-1}\varphi\|_{\mathcal{P}_I^n(E)}\varepsilon_s(z) \end{aligned}$$

By Corollary 2.1.7, $\|\gamma^{n-1}\varphi\|_{\mathcal{P}_I^n(E)} \leq e\|\gamma\|^{n-1}\|\varphi\|$, so $\Psi_{\gamma^{n-1}}$ is continuous and

$$\|\Psi_{\gamma^{n-1}}\| \leq e\|\gamma\|^{n-1}.$$

Take now $u \in \mathfrak{A}(E,F)$. Since $T_{\gamma^{n-1}u} = u \circ \Psi_{\gamma^{n-1}}$, we have $T_{\gamma^{n-1}u} \in \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$. Therefore, $\gamma^{n-1}u \in \mathfrak{A}_n(E,F)$ and

$$\|\gamma^{n-1}u\|_{\mathfrak{A}_n(E,F)} \leq \|u\|_{\mathfrak{A}(E,F)}\|\Psi_{\gamma^{n-1}}\| \leq e\|\gamma\|^{n-1}\|u\|_{\mathfrak{A}(E,F)},$$

from which (ii) follows, with constant $B = e$. □

The previous proposition gives a simple way to obtain a great variety of compatible polynomial ideals from a fixed operator ideal. Recall that, on the other hand, given a polynomial ideal, there exists only one operator ideal compatible with it.

We use the results of this section to prove that the ideals $\mathcal{P}_{PI}^n$ and $\mathcal{P}_{GI}^n$ of Piestch and Grothendieck integral polynomials are compatible with the ideals of Piestch and Grothendieck integral operators.

First, recall that $\mathcal{P}_{PI}^n(E,F) = \mathcal{L}_{PI}(\bigotimes_{\varepsilon_s}^{n,s} E, F)$ and $\mathcal{P}_{GI}^n(E,F) = \mathcal{L}_{GI}(\bigotimes_{\varepsilon_s}^{n,s} E, F)$ isometrically [CL05, Vil03].

Corollary 2.5.2. *The ideals $\mathcal{P}_{PI}^n$ and $\mathcal{P}_{GI}^n$ are compatible with the ideals of Piestch and Grothendieck integral operators respectively, with constants $A = 1$ and $B = e$.*

Proof. By Proposition 2.5.1 we have the compatibility with constants $A = B = e$. But in this case it is easy to see that we can take $A = 1$. Indeed, let $\Phi_{a^{n-1}}$ be the operator defined in the proof of Proposition 2.5.1 and take $x \in E$. Then

$$\varepsilon_s\big(\Phi_{a^{n-1}}(x), \bigotimes^{n,s} E\big) = \varepsilon_s\big(\sigma(a \otimes \cdots \otimes a \otimes x), \bigotimes^{n,s} E\big) = \sup_{\gamma \in B_{E'}} |\gamma(a)^{n-1}\gamma(x)| \leq \|a\|^{n-1}\|x\|.$$

Therefore $\|\Phi_{a^{n-1}}\| = \|a\|^{n-1}$. □

If $\mathfrak{A}$ is a maximal operator ideal, by the representation theorem [DF93, Section 17] there exists a finitely generated (2-fold) tensor norm β such that:

$$\begin{aligned} \mathfrak{A}(E,F) &\overset{1}{=} (E \otimes_\beta F')' \cap \mathcal{L}(E,F) \\ \mathfrak{A}(E,F') &\overset{1}{=} (E \otimes_\beta F)' \end{aligned}$$

In this case, we write $\mathfrak{A} = \mathcal{L}_\beta$ and say that $\mathfrak{A}$ is dual to the tensor norm β. Floret [Flo01] extends these concepts to the polynomial setting with the introduction of mixed tensor norms. We recall his definitions:

Definition 2.5.3. (i) A *mixed tensor norm* δ of order $n+1$ is an assignment of a norm on $\bigotimes^{n,s} E \otimes F$ to each pair (E,F) such that

(a) $\delta(\bigotimes^{n,s} 1 \otimes 1, \bigotimes^{n,s} \mathbb{C} \otimes \mathbb{C}) = 1$

(b) δ satisfies the metric mapping property.

(ii) A polynomial ideal $\mathfrak{A}_n$ is dual to the tensor norm δ (we write $\mathfrak{A}_n = \mathcal{P}_\delta^n$) if for every E, F,

$$\begin{aligned} \mathfrak{A}_n(E,F) &\overset{1}{=} (\bigotimes^{n,s} E \otimes F', \delta)' \cap \mathcal{P}^n(E,F) \\ \mathfrak{A}_n(E,F') &\overset{1}{=} (\bigotimes^{n,s} E \otimes F, \delta)' \end{aligned}$$

A polynomial ideal $\mathfrak{A}_n$ is maximal if and only if it is dual to a finitely generated mixed tensor norm δ (see [Flo01, 7.8]).

Following [FH02], if α is a symmetric n-tensor norm and β is a 2-fold tensor norm, we denote (α, β) the mixed tensor norm on $\bigotimes^{n,s} E \otimes F$ given by

$$\Big(\bigotimes^{n,s} E \otimes F, (\alpha, \beta)\Big) \overset{1}{=} \bigotimes_{\alpha}^{n,s} E \otimes_\beta F.$$

If $\mathfrak{A} = \mathcal{L}_\beta$ and $\mathfrak{A}_n$ is the polynomial ideal given by $\mathfrak{A}_n(E, F) = \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$, it follows that $\mathfrak{A}_n$ is dual to (α, β). In particular, if α and β are finitely generated, $\mathfrak{A}_n$ is maximal.

In [FH02, 4.2], it is conjectured that not every maximal ideal is dual to a mixed tensor norm of the form (α, β). We now show that this conjecture is true, presenting a maximal polynomial ideal that is not dual to any (α, β) norm. First we need the following result, which is of independent interest:

Proposition 2.5.4. *Let $\mathfrak{A}$ be an operator ideal and $\mathfrak{A}_n$ an ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$. If $\mathfrak{A}_n = \mathcal{P}^n_{(\alpha,\beta)}$ for some 2-fold tensor norm β and some symmetric n-tensor norm α, then:*

a) $\mathfrak{A} = \mathcal{L}_\beta$;

b) $\mathfrak{A}_n(E, F) = \mathfrak{A}(\bigotimes_\alpha^{n,s} E, F)$.

Proof. We have $\mathfrak{A}_n(E, F) = \big(\bigotimes_\alpha^{n,s} E \otimes_\beta F'\big)' \cap \mathcal{P}^n(E, F) = \mathcal{L}_\beta(\bigotimes_\alpha^{n,s} E, F)$. Then, by Proposition 2.5.1, $\mathfrak{A}_n$ is compatible with $\mathcal{L}_\beta$. By uniqueness of the compatible operator ideal (Proposition 2.1.5), $\mathfrak{A} = \mathcal{L}_\beta$. □

In [CDSP07] it is shown that the ideal of r-dominated n-linear forms is maximal and a finitely generated n-tensor norm is presented to which it is dual. Using the same ideas we prove an analogous statement for vector-valued polynomials.

For $z \in \bigotimes^{n,s} E \otimes F$ we define

$$\delta_r^n(z, \bigotimes^{n,s} E \otimes F) = \inf\Big\{ w_r((x_i)_{i=1}^m)^n \ell_u((y_i)_{i=1}^m) : \; z = \sum_{i=1}^m x_i^n \otimes y_i \Big\},$$

where $\frac{1}{u} + \frac{n}{r} = 1$ and $\ell_u((y_i)_{i=1}^m) = (\sum_{i=1}^m \|y_i\|^u)^{\frac{1}{u}}$.

Proceeding as in [DF93, 12.5] it can be seen that δ_r^n is a finitely generated mixed tensor norm. Also, we have:

Lemma 2.5.5. *$\mathcal{D}_r^n$ is dual to the mixed tensor norm δ_r^n. In particular, $\mathcal{D}_r^n$ is a maximal polynomial ideal.*

Proof. We show that $\mathcal{D}_r^n(E, F') = (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$. In a similar way it can be proved that $\mathcal{D}_r^n(E, F) = (\bigotimes^{n,s} E \otimes F', \delta_r^n)' \cap \mathcal{P}^n(E, F)$.

Let $P \in \mathcal{D}_r^n(E, F')$ and $z \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)$. For any representation $z = \sum_{i=1}^m x_i^n \otimes y_i$, we have

$$\begin{aligned} |\langle P, z\rangle| &= \Big|\sum_{i=1}^m P(x_i)(y_i)\Big| \leq \ell_u((y_i)_{i=1}^m)\ell_{\frac{r}{n}}((P(x_i))_{i=1}^m) \\ &\leq \|P\|_{\mathcal{D}_r^n(E,F')} w_r((x_i)_{i=1}^m)^n \ell_u((y_i)_{i=1}^m). \end{aligned}$$

This is true for any representation of z, thus $P \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$ and $\|P\|_{(\bigotimes^{n,s} E \otimes F, \delta_r^n)'} \leq \|P\|_{\mathcal{D}_r^n(E,F')}$.

Conversely, let $P \in (\bigotimes^{n,s} E \otimes F, \delta_r^n)'$, choose $\varepsilon > 0$ and a sequence $(x_i)_{i=1}^m \subset E$. Then for each $i = 1, \dots, m$ there exists an element $y_i \in B_F$, such that $\|P(x_i)\| \leq P(x_i)(y_i) + \frac{\varepsilon}{m}$. Also, we can find a sequence $(\lambda_i)_{i=1}^m$ of positive numbers, with $\ell_u\big((\lambda_i)_{i=1}^m\big) = 1$, such that $\sum_{i=1}^m \lambda_i(P(x_i)(y_i) + \frac{\varepsilon}{m}) = \ell_{\frac{r}{n}}\Big(\big(P(x_i)(y_i) + \frac{\varepsilon}{m}\big)_{i=1}^m\Big)$. Then

$$\begin{aligned}
\left(\sum_{i=1}^m \|P(x_i)\|^{\frac{r}{n}}\right)^{\frac{n}{r}} &\leq \left(\sum_{i=1}^m \left(P(x_i)(y_i) + \frac{\varepsilon}{m}\right)^{\frac{r}{n}}\right)^{\frac{n}{r}} \\
&= \sum_{i=1}^m \lambda_i \left(P(x_i)(y_i) + \frac{\varepsilon}{m}\right) \\
&\leq \Big\langle P, \sum_{i=1}^m x_i^n \otimes \lambda_i y_i \Big\rangle + \varepsilon \\
&\leq \|P\|_{(\bigotimes^{n,s} E\otimes F, \delta_r^n)'} w_r((x_i)_{i=1}^m)^n \ell_u((\lambda_i y_i)_{i=1}^m) + \varepsilon
\end{aligned}$$

Since $\ell_u((\lambda_i y_i)_{i=1}^m) \leq 1$ and this is valid for any $\varepsilon > 0$, we have that $P \in \mathcal{D}_r^n(E, F')$ and $\|P\|_{\mathcal{D}_r^n(E,F')} \leq \|P\|_{(\bigotimes^{n,s} E\otimes F, \delta_r^n)'}$. □

Corollary 2.5.6. *For any $n \geq 2$, the ideal of r-dominated n-homogeneous polynomials $\mathcal{D}_r^n$ is maximal but is not dual to any mixed tensor norm of the form (α, β).*

Proof. We have seen that $\mathcal{D}_r^n$ is maximal in the previous lemma. Suppose that there exists n such that $\mathcal{D}_r^n = \mathcal{P}_{(\alpha,\beta)}^n$, for some 2-fold tensor norm β and some symmetric n-tensor norm α.

Since the ideal of r-dominated polynomials is compatible with the ideal of absolutely r-summing operators, the previous proposition would assure that $\mathcal{D}_r^n(E, F) = \Pi_r(\bigotimes_\alpha^{n,s} E, F)$. Now consider $E = F = \ell_1$ and $P \in \mathcal{P}^n(\ell_1, \ell_1)$ given by

$$P(x) = \sum_{j=1}^\infty x_j^n e_j.$$

Since P factors through the absolutely 1-summing inclusion $\ell_1 \hookrightarrow \ell_2$, P is r-dominated for all $r \geq 1$ (in particular, for $r \geq n$). However, we have that $T_P(e_j^n) = e_j$, and therefore $T_P \in \mathcal{L}(\bigotimes_\alpha^{n,s} \ell_1, \ell_1)$ cannot be weakly compact (independently of the choice of α). Consequently, T_P is not absolutely r-summing, which leads to a contradiction. Therefore, there is no α and β such that $\mathcal{D}_r^n = \mathcal{P}_{(\alpha,\beta)}^n$. □

Note that the previous corollary also gives the following:

Corollary 2.5.7. *There are mixed tensor norms that are not equivalent to any (α, β)-norm.*

Since there are mixed tensor norms that are not of the form (α, β), it is now desirable to point out conditions on a mixed tensor norm (or a sequence of mixed tensor norms) that ensure compatibility (or coherence) with an operator ideal.

For $a \in E$ and $\gamma \in E'$ we define the following mappings:

$$\begin{aligned}
\Phi_{a^{n-1}}^F : E \otimes F &\to \bigotimes{}^{n,s} E \otimes F \\
x \otimes y &\mapsto \sigma(a^{n-1} \otimes x) \otimes y
\end{aligned}
\qquad
\begin{aligned}
\Psi_{\gamma^{n-1}}^F : \bigotimes{}^{n,s} E \otimes F &\to E \otimes F \\
x^n \otimes y &\mapsto \gamma(x)^{n-1} x \otimes y
\end{aligned}$$

Note that each of the conditions (i) and (ii) of Definition 2.1.2 can be seen as dual to continuity properties of the mappings defined above when we consider F' instead of F. This allows us to state the following result.

Proposition 2.5.8. *Let β be a 2-fold tensor norm and δ a mixed tensor norm of order $n+1$. Then, $\mathcal{P}^n_\delta$ is compatible with $\mathcal{L}_\beta$ (with constants A and B) if and only if the mappings $\Phi^{F'}_{a^{n-1}}$ and $\Psi^{F'}_{\gamma^{n-1}}$ are β-to-δ and δ-to-β continuous for every E and F, (with $\|\Phi^{F'}_{a^{n-1}}\| \leq A\|a\|^{n-1}$ and $\|\Psi^{F'}_{\gamma^{n-1}}\| \leq B\|\gamma\|^{n-1}$).*

2.6 Maximal, minimal and adjoint ideals

In this section we show that compatibility is preserved by some other natural procedures usually applied to polynomial ideals. First we consider the adjoint of a polynomial ideal. We show how to construct the adjoint of a vector-valued normed polynomial ideal. This is vector-valued version of [Flo01, 4.3] (see also the Preliminaries 1.2.1).

Let $\mathfrak{A}_n$ be a normed ideal of n-homogeneous polynomials. For each pair of Banach spaces E and F, we define the adjoint ideal $\mathfrak{A}_n^*(E,F)$ as a vector-valued version of [Flo01, 4.3]: for $M \in FIN(E)$, $N \in FIN(F)$ we define λ the mixed tensor norm of order $n+1$ given by

$$\Big(\bigotimes^{n,s} M \otimes N, \lambda\Big) \stackrel{1}{=} \mathfrak{A}_n(M', N).$$

That is, $z = \sum_i x_i^n \otimes y_i \in \bigotimes^{n,s} M \otimes N$ is associated to $P^z \in \mathfrak{A}_n(M', N)$, where $P^z(x') = \sum_i x'(x_i)^n y_i$; and $\lambda(z; \bigotimes^{n,s} M \otimes N) := \|P^z\|_{\mathfrak{A}_n(M',N)}$.

For $z \in \bigotimes^{n,s} E \otimes F$, we define

$$\lambda(z; \bigotimes^{n,s} E \otimes F) := \inf\Big\{\lambda(z; \bigotimes^{n,s} M \otimes N)\Big\},$$

where the infimum is taken over all $M \in FIN(E)$, $N \in FIN(F)$ such that $z \in \bigotimes^{n,s} M \otimes N$. Finally, the adjoint ideal $\mathfrak{A}_n^*$ is

$$\mathfrak{A}_n^*(E,F) := \Big(\bigotimes^{n,s} E \otimes F', \lambda\Big)' \bigcap \mathcal{P}^n(E,F).$$

Proposition 2.6.1. *Let $\mathfrak{A}$ be a normed operator ideal. If $\mathfrak{A}_n$ is a normed ideal of n-homogeneous polynomials compatible with $\mathfrak{A}$ with constants A and B, then $\mathfrak{A}_n^*$ is compatible with $\mathfrak{A}^*$ with constants B and A.*

The proof is analogous to Proposition 3.1.30.

To take maximal hulls of the ideals $\mathfrak{A}$ and $\mathfrak{A}_n$ also preserves the compatibility. Indeed, since $\mathfrak{A}_n^{\max}$ coincides with $\mathfrak{A}_n^{**}$, we have the following:

Corollary 2.6.2. *If $\mathfrak{A}_n$ is compatible with $\mathfrak{A}$ with constants A and B, then $\mathfrak{A}_n^{\max}$ is compatible with $\mathfrak{A}^{\max}$ with constants A and B.*

The results of this section allows us to obtain a different proof of the compatibility of the ideal of Grothendieck integral polynomials. Since $\mathcal{P}^n_{GI} = (\mathcal{P}^n_N)^{\max}$, it is an immediate consequence of Corollary 2.6.2, and Example 2.1.9 that the Grothendieck integral polynomials are compatible with the ideal of Grothendieck integral operators with constants $A = 1$ and $B = e$. The same conclusion follows using that $\mathcal{P}^n_{GI} = (\mathcal{P}^n)^*$ and Proposition 2.6.1 and Example 2.1.8.

The reciprocal of Proposition 2.6.1 and Corollary 2.6.2 is false in general, see Remark 3.1.33.

Now we consider the minimal hull of a polynomial ideal. Since $\overline{\mathcal{F}}$ is a closed operator ideal, the following corollary is just a particular case of Proposition 2.3.1.

Corollary 2.6.3. *If $\mathfrak{A}_n$ is a Banach polynomial ideal compatible with a Banach operator ideal $\mathfrak{A}$, with constants A and B, then $\mathfrak{A}_n^{\min}$ is compatible with $\mathfrak{A}^{\min}$ with constants A and B.*

Finally, we relate the results of this section with the largest and smallest compatible polynomial ideals defined in Section 2.2.

Proposition 2.6.4. *Let $\mathfrak{A}$ be a normed operator ideal. Then:*
(i) $\big(\mathcal{M}_n^{\mathfrak{A}}\big)^{\max} \overset{1}{=} \big(\mathcal{M}_n^{\mathfrak{A}^{\max}}\big)$;
(ii) $\big(\mathcal{N}_n^{\mathfrak{A}}\big)^{\min} \overset{1}{=} \big(\mathcal{N}_n^{\mathfrak{A}^{\min}}\big)$.

Proof. We only prove (i) (the proof of (ii) is similar).

Since $\mathcal{M}_n^{\mathfrak{A}}$ is compatible with $\mathfrak{A}$ with constants $A = B = 1$, its maximal hull $\big(\mathcal{M}_n^{\mathfrak{A}}\big)^{\max}$ is also compatible with $\mathfrak{A}^{\max}$ with constants $A = B = 1$ by Corollary 2.6.2. Therefore $\big(\mathcal{M}_n^{\mathfrak{A}}\big)^{\max} \subset \mathcal{M}_n^{\mathfrak{A}^{\max}}$, and by Proposition 2.2.5 the inclusion has norm 1.

Conversely, for $M, N \in FIN$ we have $\mathfrak{A}(M,N) \overset{1}{=} \mathfrak{A}^{\max}(M,N)$. As a consequence, $\mathcal{M}_n^{\mathfrak{A}}(M,N) = \mathcal{M}_n^{\mathfrak{A}^{\max}}(M,N)$ isometrically, which implies that $\mathcal{M}_n^{\mathfrak{A}^{\max}} \subset \big(\mathcal{M}_n^{\mathfrak{A}}\big)^{\max}$ with norm one inclusion. $\square$

Thus, the last proposition states in particular that if $\mathfrak{A}$ is a maximal (respectively minimal) normed operator ideal then $\mathcal{M}_n^{\mathfrak{A}}$ (respectively $\mathcal{N}_n^{\mathfrak{A}}$) is maximal (respectively minimal).

We also have the following result that establishes a duality relationship between the smallest and largest Banach polynomial ideals compatible with an operator ideal.

Proposition 2.6.5. *Let $\mathfrak{A}$ be a normed operator ideal. Then $\big(\mathcal{N}_n^{\mathfrak{A}}\big)^* = \mathcal{M}_n^{\mathfrak{A}^*}$.*

Proof. Let $M, N \in FIN$. Then

$$\begin{aligned} \big(\mathcal{N}_n^{\mathfrak{A}}\big)^*(M,N) &= \mathcal{N}_n^{\mathfrak{A}}(M',N')' \\ &= \mathcal{P}_N^{n-1}\big(M', \mathfrak{A}(M',N')\big)' \\ &= \mathcal{P}^{n-1}\big(M, \mathfrak{A}(M',N')'\big) \\ &= \mathcal{P}^{n-1}\big(M, \mathfrak{A}^*(M,N)\big) \\ &= \mathcal{M}_n^{\mathfrak{A}^*}(M,N), \end{aligned}$$

where second and last identities follow from Remark 2.2.1 and all equalities are isometric.

Since every adjoint ideal is maximal, we have $\big(\mathcal{N}_n^{\mathfrak{A}}\big)^* = \big(\mathcal{M}_n^{\mathfrak{A}^*}\big)^{\max}$. Therefore, $\big(\mathcal{M}_n^{\mathfrak{A}^*}\big)^{\max} = \mathcal{M}_n^{\mathfrak{A}^*}$, by Proposition 2.6.4 (see the above comments). $\square$

2.7 Some applications

Suppose that we know how to characterize some property of a Banach space E in terms of linear operators on (or into) it. We now show how the notion of compatibility may be useful to generalize this kind of characterization in terms of homogeneous polynomials on (or into) E.

Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$

Lewis and Stegall [LS73] characterized Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$ for some set Γ in terms of nuclear and absolutely summing operators on E. Specifically they proved

Theorem 2.7.1. *Let E be a Banach space then the following are equivalent:*
(i) E' is isomorphic to $\ell_1(\Gamma)$ for some set Γ.
(ii) For every Banach space F, $\Pi_1(E,F) \subset \mathcal{L}_N(E,F)$.
(iii) For every Banach space F, $\Pi_1(E,F) \cap \mathcal{L}_K(E,F) \subset \mathcal{L}_N(E,F)$.

This characterization was generalized by Cilia, D'Anna and Gutierrez [CDG04], using 1-dominated polynomials. We will now show this result and some similar characterizations using the concept of compatibility (see also [BP05] for more results in this direction).

Theorem 2.7.2. *Let E be a Banach space then the following are equivalent:*
(i) E' is isomorphic to $\ell_1(\Gamma)$ for some set Γ.
(ii) For every Banach space F, $\mathcal{F}_n^{\Pi_1}(E,F) \subset \mathcal{P}_N^n(E,F)$, for every (or for some) n.
(iii) For every Banach space F, $\mathcal{F}_n^{\Pi_1}(E,F) \cap \mathcal{F}_n^{\mathcal{L}_K}(E,F) \subset \mathcal{P}_N^n(E,F)$, for every (or for some) n.
(iv) For every Banach space F, $\mathcal{D}_1^n(E,F) \subset \mathcal{P}_N^n(E,F)$, for every (or for some) n.
(v) For every Banach space F, $\mathcal{D}_1^n(E,F) \cap \mathcal{P}_K^n(E,F) \subset \mathcal{P}_N^n(E,F)$, for every (or for some) n.

Proof. $(i) \Rightarrow (ii)$: by the theorem of Lewis and Stegall above, we have that $\Pi_1(E,F) \subset \mathcal{L}_N(E,F)$ for every Banach space F. By Proposition 2.2.4 we have that $\mathcal{F}_n^{\Pi_1}(E,F) \subset \mathcal{P}_N^{n-1}(E,\mathcal{L}_N(E,F)) = \mathcal{P}_N^n(E,F)$.
$(ii) \Rightarrow (iii)$ is clear.
$(iii) \Rightarrow (i)$: by Proposition 2.1.5 and the theorem of Lewis and Stegall.
$(i) \Rightarrow (iv)$: it shown in [Sch91] that $\mathcal{D}_1^n(E,F) = \mathcal{P}^n \circ \Pi_1(E,F)$. On the other hand, $\mathcal{P}^n \circ \mathcal{L}_N(E,F) \subset \mathcal{P}_N(E,F)$, indeed if $T = \sum \gamma_j y_j \in \mathcal{L}_N(E,G)$, $\|\gamma_j\|_{E'} = 1$, $(\|y_j\|_G)_j \in \ell_1$ and $Q \in \mathcal{P}^n(G,F)$ then

$$\begin{aligned} P(x) &= QT(x) = Q(\sum \gamma_j(x) y_j) = \sum_{j_1,\dots,j_n} \gamma_{j_1} \dots \gamma_{j_n} \overset{\vee}{Q}(y_{j_1},\dots,y_{j_n}) \\ &= \frac{1}{2^n n!} \sum_{j_1,\dots,j_n} \sum_{\varepsilon_1,\dots,\varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n (\varepsilon_1 \gamma_{j_1} + \dots + \varepsilon_n \gamma_{j_n})^n \overset{\vee}{Q}(y_{j_1},\dots,y_{j_n}). \end{aligned}$$

Since $(\|\varepsilon_1 \gamma_{j_1} + \dots + \varepsilon_n \gamma_{j_n}\|^n)_{j_1,\dots,j_n} \in \ell_\infty$ and $(\|Q(y_{j_1},\dots,y_{j_n})\|)_{j_1,\dots,j_n} \in \ell_1$, we conclude that P is nuclear (this was proved in [CDG04, Proposition 4]). We know that $\Pi_1(E,F) \subset \mathcal{L}_N(E,F)$ for every Banach space F, and this implies easily that $\mathcal{D}_1^n(E,F) = \mathcal{P}^n \circ \Pi_1(E,F) \subset \mathcal{P}^n \circ \mathcal{L}_N(E,F) \subset \mathcal{P}_N(E,F)$ for every Banach space F.
$(iv) \Rightarrow (v)$ is clear.
$(v) \Rightarrow (i)$ by Proposition 2.1.5 and the theorem of Lewis and Stegall. □

Asplund spaces

Alencar proved in [Ale85a, Theorem 1.3] that a Banach space is Asplund if and only if $\mathcal{L}_{PI}(E,F) \subset \mathcal{L}_N(E,F)$ for every Banach space F and in that case $\mathcal{L}_{PI}(E,F) \overset{1}{=} \mathcal{L}_N(E,F)$. In [Ale85b] he proved that if E is Asplund then $\mathcal{P}_{PI}^n(E,F) = \mathcal{P}_N^n(E,F)$ with equivalent norms for every Banach space F and every n (the isometry was proved by Carando and Dimant in [CD00, Theorem 1.4]). The converse (which was first showed by Cilia and Gutierrez in [CG04], see also [CG05] and [BP05]) is an easy Corollary of those results and Proposition 2.1.5.

Corollary 2.7.3. *E is Asplund if and only if for some $n \in \mathbb{N}$, $\mathcal{P}_{PI}^n(E,F) \subset \mathcal{P}_N^n(E,F)$ for every Banach space F.*

Proof. We have already seen in the examples that $\mathcal{P}^n_{PI}$ and $\mathcal{P}^n_N$ are compatible with $\mathcal{L}_{PI}$ and $\mathcal{L}_N$ respectively. Thus $\mathcal{P}^n_{PI}(E,F) \subset \mathcal{P}^n_N(E,F)$ for every Banach space F implies by Proposition 2.1.5 that $\mathcal{L}_{PI}(E,F) \subset \mathcal{L}_N(E,F)$. Therefore by [Ale85a, Theorem 1.3], E is Asplund. The converse is proved in [Ale85b]. □

Note that we can arrive to the same conclusion using any polynomial ideals compatible with $\mathcal{L}_{PI}$ and $\mathcal{L}_N$.

$\mathcal{L}_\infty$-spaces

Stegall and Retheford [SR72] showed that a Banach space E is an $\mathcal{L}_\infty$-space if and only if every absolutely 1-summing operator on E is Grothendieck integral. Cilia, D'Anna and Gutierrez [CDG02] proved a similar characterization but using 1-dominated homogeneous polynomials. They proved that a Banach space E is an $\mathcal{L}_\infty$-space if and only if every 1-dominated n-homogeneous polynomial from E to F is G-integral, for every F and some n. One implication ([CDG02, Proposition 3.1]) is also a corollary of Proposition 2.1.5.

Corollary 2.7.4. *If for some $n \in \mathbb{N}$, $\mathcal{D}^n_1(E,F) \subset \mathcal{P}^n_{GI}(E,F)$ for every Banach space F then E is an $\mathcal{L}_\infty$-space.*

Proof. We have already seen in the examples that $\mathcal{P}^n_{GI}$ and $\mathcal{D}^n_1$ are compatible with $\mathcal{L}_{GI}$ and Π_1 respectively. Thus $\mathcal{D}^n_1(E,F) \subset \mathcal{P}^n_{GI}(E,F)$ for every Banach space F implies by Proposition 2.1.5 that $\mathcal{D}_1(E,F) \subset \mathcal{L}_{GI}(E,F)$. Therefore by [SR72], E is an $\mathcal{L}_\infty$-space. □

Chapter 3

Coherent sequences of polynomial ideals and holomorphic mappings of bounded type

In this chapter we relate sequences of polynomial ideals of different degrees. We define the concept of coherence of a sequence of homogeneous polynomials of different degrees and study some of their basic properties. We associate to each coherent sequence a Fréchet space of entire mappings of bounded type.

For the case of scalar valued functions we study convolution operators. A result of Godefroy and Shapiro states that the convolution operators on the space of entire functions on $\mathbb{C}^n$, which are not multiples of identity, are hypercyclic. We determine conditions on the coherent sequence that assure the hypercyclicity of convolution operators on spaces of holomorphic functions on a Banach space. Some known results come out as particular cases of this setting. We also consider holomorphic functions associated to minimal ideals of polynomials and to polynomials of the Schatten-von Neumann classes.

We also study spaces of holomorphic mappings associated to coherent sequences on balls and more general open sets. Most of the content of this chapter belong to the articles [CDM07, CDM09, CDM].

3.1 Coherent sequences

In this chapter we relate sequences of polynomial ideals of different degrees. Our goal is to define holomorphic functions of a given type as a series of homogeneous polynomials pertaining to ideals which are related to each other, much in the spirit of holomorphy types defined by Nachbin [Nac69].

As in the previous chapter, the relationship between the ideals of polynomials in the sequence is given by the operations of fixing a variable or multiplying by a linear functional, motivated by Proposition 2.1.1.

A difference with the previous chapter is that compatibility has no sense for scalar ideals. This is because the only scalar ideal of 1-homogeneous polynomials is the ideal of all continuous linear forms, thus every scalar ideal of homogeneous polynomials is compatible with it.

Definition 3.1.1. *Consider the sequence $\{\mathfrak{A}_k\}_{k=1}^N$, where for each k, $\mathfrak{A}_k$ is a quasi-normed ideal of k-homogeneous polynomials and N is eventually infinite. We say that $\{\mathfrak{A}_k\}_k$ is a* ***coherent***

sequence of polynomial ideals *if there exist positive constants C and D such that for every Banach spaces E and F, the following conditions hold for $k = 1, \dots, N-1$:*
(i) For each $P \in \mathfrak{A}_{k+1}(E, F)$ and $a \in E$, P_a belongs to $\mathfrak{A}_k(E; F)$ and

$$\|P_a\|_{\mathfrak{A}_k(E,F)} \leq C\|P\|_{\mathfrak{A}_{k+1}(E,F)}\|a\|$$

(ii) For each $P \in \mathfrak{A}_k(E, F)$ and $\gamma \in E'$, γP belongs to $\mathfrak{A}_{k+1}(E, F)$ and

$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E,F)} \leq D\|\gamma\|\|P\|_{\mathfrak{A}_k(E,F)}$$

Let $\mathfrak{A}$ be a linear operator ideal. We say that the sequence of k-homogeneous polynomial ideals $\{\mathfrak{A}_k\}_k$ is a ***coherent sequence associated to*** *$\mathfrak{A}$ if $\{\mathfrak{A}_k\}_k$ is a coherent sequence and $\mathfrak{A}_1 = \mathfrak{A}$.*

Also, we say that the sequence of k-homogeneous polynomial ideals $\{\mathfrak{A}_k\}_k$ is ***coherent at*** *(E, F) (or simply at E when $F = \mathbb{C}$) if conditions (i) and (ii) are fulfilled for fixed Banach spaces E and F.*

Note that if $\{\mathfrak{A}_k\}_{k=1}^N$ is a coherent sequence, then for each $k = 1, \dots, N$, the polynomial ideal $\mathfrak{A}_k$ is compatible with $\mathfrak{A} = \mathfrak{A}_1$ with constants $A \leq C^{k-1}$ and $B \leq D^{k-1}$. Nevertheless, in most of the natural examples one obtains better estimates.

As mentioned before, our definitions of compatibility and coherence are related to other concepts studied by several authors. Indeed, property (i) in Definition 2.1.2 implies the polynomial ideal to be closed under differentiation; property (ii) in Definition 3.1.1 implies that the polynomial ideal is closed for scalar multiplication (see [BP05]) and property (i) in Definition 3.1.1 is what in [BBJP06] is called "polynomial property (B)". Also, coherent sequences are always global holomorphy types.

Although we are working with complex Banach spaces, it is clear that Definition 3.1.1 (and also Definition 2.1.2) make sense for polynomial ideals on real Banach spaces. However, Botelho, Braunss, Junek and Pellegrino showed that in the real case, no sequence of closed polynomial ideals is coherent, in particular the sequence of ideals of all polynomials is not coherent:

Proposition 3.1.2. *[BBJP06, Proposition 8.5] For real Banach spaces, there exists no coherent sequence of closed polynomial ideals. Specifically, given any sequence of closed polynomial ideals, there exists no constant $C > 0$ such that condition (i) in Definition 3.1.1 is satisfied.*

Proof. Suppose that such a $c > 0$ exists. In [Har97], Harris defines the constants $c_{n,n-1}$, as the infimum of all constants K which satisfy that

$$\|d^{n-1}P\| = n! \sup_{\|a\|=1} \|P_a\| \leq K\|P\|,$$

and shows that there exist n-homogeneous polynomials $P_n \in \mathcal{P}^n((\mathbb{R}^2, \|\cdot\|_1))$ such that

$$n! \sup_{\|a\|=1} \|(P_n)_a\| = c_{n,n-1}\|P_n\|.$$

Therefore there exist $a_n \in B_{\mathbb{R}^2}$ such that $n!\|(P_n)_{a_n}\| = c_{n,n-1}\|P_n\|$.

On the other hand, Revesz and Sarantopoulos [RS03] proved that there exist a constant $K > 0$ such that $c_{n,n-1} \geq Kn! \ln n$ for every $n \in \mathbb{N}$. Thus, $Kn! \ln n\|P_n\| \leq c_{n,n-1}\|P_n\| = n!\|(P_n)_{a_n}\| \leq cn!\|P_n\|$ for every n, which is a contradiction. ☐

Therefore, the concept of coherence for real Banach spaces is too restrictive, since the most natural sequence of polynomial ideals fails to fulfill it. Note that even in this case, the ideal of all polynomials of a fixed degree is compatible with the ideal of all operators. This means that the concept of compatibility could be interesting also in the real case.

The following lemma show a kind of converse to conditions *(i)* and *(ii)* of Definition 3.1.1.

Lemma 3.1.3. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of normed ideals of homogeneous polynomials and $P \in \mathcal{P}^k(E,F)$. Then the following are equivalent:*

a) $P \in \mathfrak{A}_k(E,F)$.

b) γP belongs to $\mathfrak{A}_{k+1}(E,F)$ for all $\gamma \in E'$ (or for some nonzero $\gamma \in E'$).

c) For each $0 \neq a \in E$, there exists $Q \in \mathfrak{A}_{k+1}(E,F)$ such that $P = Q_a$.

d) There exists $Q \in \mathfrak{A}_{k+1}(E,F)$ and $a \in E$ such that $P = Q_a$.

Proof. *a)* $\Rightarrow$ *b)*, *c)* $\Rightarrow$ *a)* and *d)* $\Rightarrow$ *a)* by the definition of coherence. *c)* $\Rightarrow$ *d)* is obvious.

b) $\Rightarrow$ *a)*: Let $R = \gamma P \in \mathfrak{A}_{k+1}(E,F)$ and choose $a \in E$ such that $\gamma(a) = 1$. Proceeding as in the proof of [AS76, Proposition 5.3], we can write P as

$$P = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^{j-1} R_{a^j}. \tag{3.1}$$

By the coherence of $\{\mathfrak{A}_k\}_k$, $\gamma^{j-1} R_{a^j}$ belongs to $\mathfrak{A}_k(E,F)$ for each j and we conclude that so does P.

a) $\Rightarrow$ *c)*: Let $P \in \mathfrak{A}_k(E,F)$ and $0 \neq a \in E$ and take $\gamma \in E'$ such that $\gamma(a) = 1$. It is not difficult to check that, for each $j = 1, \dots, n+1$, we have $\big(\gamma^j P_{a^{j-1}}\big)_a = \gamma^{j-1} \big(\gamma P\big)_{a^j}$. Then, by equation (3.1) we obtain

$$P = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \big(\gamma^j P_{a^{j-1}}\big)_a = \left(\sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^j P_{a^{j-1}} \right)_a .$$

Clearly, $Q = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} \gamma^j P_{a^{j-1}}$ belongs to $\mathfrak{A}_{k+1}(E,F)$ by the coherence of the sequence $\{\mathfrak{A}_k\}$. $\square$

We present a result which is the analogous to Proposition 2.1.5 for coherent sequences of polynomial ideals (see also [BP05, BBJP06]).

Proposition 3.1.4. *Let $\{\mathfrak{A}_k\}_k$ and $\{\mathfrak{B}_k\}_k$ be coherent sequences. If for some E and F and some k_0, $\mathfrak{A}_{k_0}(E,F) \subset \mathfrak{B}_{k_0}(E,F)$, then $\mathfrak{A}_k(E,F) \subset \mathfrak{B}_k(E,F)$ for all $k \leq k_0$.*

As a corollary, we obtain that there can be at most one coherent sequence $\{\mathfrak{B}_k\}_{k=1}^n$ with $\mathfrak{B}_n = \mathfrak{A}_n$.

Let us see some examples of coherent sequences.

Example 3.1.5. *Continuous homogeneous polynomials: $\mathcal{P}$.*

The sequence $\{\mathcal{P}^k\}_{k=1}^\infty$ is a coherent sequence with constants $C = e$ and $D = 1$. Both results follow from Corollary 2.1.7.

Similarly, the sequences of approximable, compact, weakly compact, weakly sequentially continuous and weakly continuous on bounded sets polynomials are coherent with the corresponding operator ideals.

Example 3.1.6. *Nuclear polynomials: $\mathcal{P}_N$*

As in Example 2.1.9 it is easy to show that the sequence of nuclear polynomials is coherent with the ideal of nuclear operators with constants $C = 1$ and $D = e$.

Example 3.1.7. *Integral polynomials: $\mathcal{P}_{PI}$ and $\mathcal{P}_{GI}$*

The sequence of integral polynomials is coherent with the ideal of integral operators, with constants $C = 1$ and $D = e$, as we will see in Section 3.1.3.

Example 3.1.8. *Extendible polynomials: $\mathcal{P}_e$*

As in Example 2.1.11 we can show that the sequence of extendible polynomials is coherent with the ideal of extendible operators, with constants $C = e$ and $D = 1$.

Example 3.1.9. *Multiple r-summing polynomials: $\mathcal{M}_r$*

We set $\mathcal{M}_r^1$ to be the ideal of absolutely r-summing operators, Π_r. Let $P \in \mathcal{M}_r^k(E,F)$. For $a \in E$, it is immediate that P_a is multiple r-summing and $\|P_a\|_{\mathcal{M}_r^{k-1}} \leq \|a\|\|P\|_{\mathcal{M}_r^k}$. Also for any $\gamma \in E'$, we have

$$(\gamma P)^\vee(x_1, \ldots, x_{k+1}) = \frac{1}{k+1} \sum_{j=1}^{k+1} \gamma(x_j) \overset{\vee}{P}(x_1, \ldots, \widetilde{x_j}, \ldots, x_{k+1})$$

where $\widetilde{x_j}$ means that this coordinate is omitted. Then, by the triangle inequality,

$$\begin{aligned}
&\left(\sum_{i_1,\ldots,i_{k+1}=1}^{m_1,\ldots,m_{k+1}} \|(\gamma P)^\vee(x_1^{i_1}, \ldots, x_{k+1}^{i_{k+1}})\|^r \right)^{\frac{1}{r}} \leq \\
&\leq \frac{1}{k+1} \sum_{j=1}^{k+1} \left(\sum_{i_1,\ldots,i_{k+1}=1}^{m_1,\ldots,m_{k+1}} |\gamma(x_j^{i_j})|^r \left\| \overset{\vee}{P}(x_1^{i_1}, \ldots, \widetilde{x_j^{i_j}}, \ldots, x_{k+1}^{i_{k+1}}) \right\|^r \right)^{1/r} \\
&= \frac{1}{k+1} \sum_{j=1}^{k+1} \left(\sum_{i_j=1}^{m_j} |\gamma(x_j^{i_j})|^r \sum_{i_1,\ldots,\widetilde{i_j},\ldots,i_{k+1}=1}^{m_1,\ldots,m_{k+1}} \left\| \overset{\vee}{P}(x_1^{i_1}, \ldots, \widetilde{x_j^{i_j}}, \ldots, x_{k+1}^{i_{k+1}}) \right\|^r \right)^{1/r} \\
&\leq \frac{1}{k+1} \sum_{j=1}^{k+1} \left(\sum_{i_j=1}^{m_j} |\gamma(x_j^{i_j})|^r \|P\|^r_{\mathcal{M}_r^k} \prod_{l=1,\ l\neq j}^{k+1} w_r((x_l^{i_l})_{i_l=1}^{m_l})^r \right)^{1/r} \\
&\leq \|\gamma\|\|P\|_{\mathcal{M}_r^k} w_r((x_1^{i_1})_{i_1=1}^{m_1}) \cdots w_r((x_{k+1}^{i_{k+1}})_{i_{k+1}=1}^{m_{k+1}}).
\end{aligned}$$

Hence, γP is multiple r-summing with $\|\gamma P\|_{\mathcal{M}_r^{k+1}} \leq \|\gamma\|\|P\|_{\mathcal{M}_r^k}$.

Therefore, $\{\mathcal{M}_r^k\}_k$ is a coherent sequence associated with the ideal of absolutely r-summing operators with constants $C = D = 1$. Consequently, compatibility constants are also $A = B = 1$ as we said in Example 2.1.12.

Example 3.1.10. *r-dominated polynomials: $\mathcal{D}_r$*

The sequence of ideals of r-dominated polynomials is coherent with the ideal of absolutely r-summing operators with constants $C = e$ and $D = 1$. We will prove this as a particular case of the composition ideals in Section 3.1.2.

As in the case of compatibility, not all the usual polynomial extensions of an operator ideal form a coherent sequence.

Example 3.1.11. *The sequence of absolutely p-summing polynomials is not coherent.*

This is a consequence of Example 2.1.23, since if the sequence were coherent then the absolutely p-summing polynomials should be compatible with the absolutely p-summing operators.

One could still wonder which should be the sequence of ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n+1}$ which is coherent with $\mathfrak{A}_{n+1} = \Pi_p^{n+1}$. A reasoning similar to the proof of 2.1.23 proves:

Example 3.1.12. The sequence $\{\mathcal{L}, \mathcal{P}^2, \ldots, \mathcal{P}^n, \Pi_p^{n+1}\}$ is coherent with constants $C = e$ and $D = 1$.

From this example and Proposition 3.1.4 we have

Corollary 3.1.13. *Suppose that $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_{n+1}\}$ is a coherent sequence and that $\Pi_p^{n+1}(E, F) \subset \mathfrak{A}_{n+1}(E, F)$. Then $\mathfrak{A}_k(E, F) = \mathcal{P}^k(E, F)$ for every $k = 1, \ldots, n$.*

In particular, every absolutely p-summing $(n + 1)$-homogeneous polynomial from E to F is weakly compact then every k-homogeneous polynomial from E to F is weakly compact, for each $k \leq n$.

To end this section we will prove that, as in the previous example, given a Banach ideal of $(n + 1)$-homogeneous polynomials $\mathfrak{A}_{n+1}$, there always exist Banach ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $\{\mathfrak{A}_k\}_{k=1}^{n+1}$ is a coherent sequence. The steps to prove this will be the same as in Subsection 2.1.1. We will state all the preliminary results, but we will just prove the few differences with the results of that subsection.

Lemma 3.1.14. *Let $\mathfrak{A}_n$ an ideal of n-homogeneous polynomials. Let $Q \in \mathcal{P}^{n-1}(E, F)$ and fix a nonzero $\gamma_0 \in E'$. Then $\gamma_0 Q \in \mathfrak{A}_n(E, F)$ if and only if $\gamma Q \in \mathfrak{A}_n(E, F)$ for every $\gamma \in E'$.*

Proof. Let $0 \neq \gamma \in E'$ and pick $a \in E$ such that $\gamma_0(a) \neq 0$. We will prove by induction that for $1 \leq k \leq n$, $\gamma\gamma_0^{n-k} Q_{a^{n-k}}$ belongs to $\mathfrak{A}_n(E, F)$. The Lemma follows taking $k = n$.

For $k = 1$, $\gamma\gamma_0^{n-1} Q_{a^{n-1}} = \gamma\gamma_0^{n-1} Q(a) \in \mathfrak{A}_n(E, F)$ since it is a finite type polynomial.

Suppose that $\gamma\gamma_0^{n-k} Q_{a^{n-k}}$ belongs to $\mathfrak{A}_n(E, F)$ (with $k < n$), then, using Lemma 2.1.17, we deduce that $\gamma\gamma_0^{n-k-1} (\gamma_0 Q)_{a^{n-k}} \in \mathfrak{A}_n(E, F)$. But

$$\gamma\gamma_0^{n-k-1} (\gamma_0 Q)_{a^{n-k}} = \alpha\gamma_0(a)\gamma\gamma_0^{n-k-1} Q_{a^{n-k-1}} + (1 - \alpha)\gamma\gamma_0^{n-k} Q_{a^{n-k}},$$

for some $\alpha \in (0, 1)$. Therefore

$$\gamma\gamma_0^{n-k-1} Q_{a^{n-k-1}} = \frac{1}{\alpha\gamma_0(a)} \Big(\gamma\gamma_0^{n-k-1} (\gamma_0 Q)_{a^{n-k}} - (1 - \alpha)\gamma\gamma_0^{n-k} Q_{a^{n-k}}\Big) \in \mathfrak{A}_n(E, F).$$

□

As we did in Section 2.1.1 we now define, for a fixed polynomial ideal $\mathfrak{A}_{n+1}$, another polynomial ideal $\mathfrak{A}_n$, and a complete norm on it.

Proposition 3.1.15. *Let $\mathfrak{A}_{n+1}$ be an ideal of $(n + 1)$-homogeneous polynomials. Define, for each pair of Banach spaces E and F,*

$$\mathfrak{A}_n(E, F) = \left\{P \in \mathcal{P}^n(E, F) / \ \gamma P \in \mathfrak{A}_{n+1}(E, F) \ \textit{for all}\ \gamma \in E'\right\},$$

with $|||P|||_{\mathfrak{A}_n(E,F)} = \sup_{\gamma \in S_{E'}} \|\gamma P\|_{\mathfrak{A}_{n+1}(E,F)}$. Then *(a)* $\mathfrak{A}_n$ *is an ideal of* n*-homogeneous polynomials and*

$$\mathfrak{A}_n(E,F) = \{Q_a \in \mathcal{P}^n(E,F) : Q \in \mathfrak{A}_{n+1}(E,F), a \in E\}.$$

(b) $|||\cdot|||_{\mathfrak{A}_n(E,F)}$ *is a norm on* $\mathfrak{A}_n(E,F)$ *and satisfies*

$$|||P|||_{\mathfrak{A}_n(E,F)} \geq \|P\|_{\mathcal{P}^n(E,F)}, \quad \textit{for every } P \in \mathfrak{A}_n(E,F).$$

Moreover, $\Big(\mathfrak{A}_n(E,F), |||\cdot|||_{\mathfrak{A}_n(E,F)}\Big)$ *is a Banach space.*

(c) $|||S \circ P|||_{\mathfrak{A}_n(E,F_1)} \leq \|S\|_{\mathcal{L}(F,F_1)} |||P|||_{\mathfrak{A}_n(E,F)}$ *for every* $S \in \mathcal{L}(F,F_1)$ *and* $P \in \mathfrak{A}_n(E,F)$*.*

(d) If E_0 *is a subspace of* E *with norm 1 inclusion* $i : E_0 \hookrightarrow E$*, then*

$$|||P \circ i|||_{\mathfrak{A}_n(E_0,F)} \leq |||P|||_{\mathfrak{A}_n(E,F)}, \quad \textit{for all } P \in \mathfrak{A}_n(E,F).$$

Proof. We will only show how to prove that $\mathfrak{A}_n(E,F) \subset \{Q_a \in \mathcal{P}^n(E,F)/\ Q \in \mathfrak{A}_{n+1}(E,F), a \in E\}$. The rest of the proof is analogous to Proposition 2.1.19.

Let $P \in \mathfrak{A}_n(E,F)$, we will show that there exists $Q \in \mathfrak{A}_{n+1}(E,F)$ and $a \in E$, such that $Q_a = P$. For every $\gamma \in E'$, we know that $\gamma P \in \mathfrak{A}_{n+1}(E,F)$. Let $a \in E$, $\gamma \in E'$ such that $\gamma(a) = 1$, and define

$$Q = \sum_{k=0}^{n} \alpha_k \gamma^{k+1} P_{a^k},$$

where $\alpha_0 = n+1$ and $(k+1)\alpha_k = -(n-k+1)\alpha_{k-1}$. Since $\mathfrak{A}_n$ is an ideal of polynomials, we have by Lemma 2.1.17 that $\gamma^k P_{a^k}$ belongs to $\mathfrak{A}_n(E,F)$ for every $k = 0, \dots, n$ and thus $\gamma^{k+1} P_{a^k} \in \mathfrak{A}_{n+1}(E,F)$. Therefore $Q \in \mathfrak{A}_{n+1}(E,F)$. An easy computation shows that $Q_a = P$. □

The following results may be proved as Propositions 2.1.20 and 2.1.21.

Proposition 3.1.16. *The norm* $|||\cdot|||_{\mathfrak{A}_n}$ *defined on Proposition 3.1.15 satisfies the "almost ideal" property: for Banach spaces* E *and* F*, there exists a constant* $c > 0$ *such that, for all Banach spaces* E_1*,* F_1 *and all operators* $R \in \mathcal{L}(E_1,E)$*,* $P \in \mathfrak{A}_n(E,F)$ *and* $S \in \mathcal{L}(F,F_1)$*, it follows that*

$$|||S \circ P \circ R|||_{\mathfrak{A}_n(E_1,F_1)} \leq c\|S\|_{\mathcal{L}(F,F_1)} |||P|||_{\mathfrak{A}_n(E,F)} \|R\|^n_{\mathcal{L}(E_1,E)}.$$

Proposition 3.1.17. *Let* $\mathfrak{A}_n$ *be an* n*-homogeneous polynomial ideal with norm* $|||\cdot|||_{\mathfrak{A}_n}$ *that satisfies the "almost ideal" property. Then we can define an equivalent norm* $\|\cdot\|_{\mathfrak{A}_n}$ *which is an ideal norm on* $\mathfrak{A}_n$*.*

Proof. Define a norm for $P \in \mathfrak{A}_n(E,F)$,

$$\|P\|'_{\mathfrak{A}_n(E,F)} = \sup\{|||S \circ P \circ R|||_{\mathfrak{A}_n(E_1,F_1)} :\ E_1, F_1 \text{ Banach spaces}, \|S\|_{\mathcal{L}(F,F_1)} = \|R\|_{\mathcal{L}(E_1,E)} = 1\}.$$

Then $\|\cdot\|'_{\mathfrak{A}_n}$ is a norm on $\mathfrak{A}_n$ equivalent to $|||\cdot|||_{\mathfrak{A}_n}$. Moreover,

$$\|S \circ P \circ R\|'_{\mathfrak{A}_n(E_1,F_1)} \leq \|S\|_{\mathcal{L}(F,F_1)} \|P\|'_{\mathfrak{A}_n(E,F)} \|R\|_{\mathcal{L}(E_1,E)}.$$

So, if we let $\kappa_n = \|z \mapsto z^n\|'_{\mathfrak{A}_n(\mathbb{C},\mathbb{C})}$, then the norm $\|\cdot\|_{\mathfrak{A}_n}$ defined by $\|P\|_{\mathfrak{A}_n(E,F)} = \frac{1}{\kappa_n}\|P\|'_{\mathfrak{A}_n(E,F)}$, is an ideal norm equivalent to $|||\cdot|||_{\mathfrak{A}_n}$. □

As in Remark 2.1.22 we can see that in the case $(\mathfrak{A}_n, |||\cdot|||_{\mathfrak{A}_n})$ is the polynomial ideal given in Proposition 3.1.15, we can simplify the definition of $\|\cdot\|'_{\mathfrak{A}_n}$:

$$\|P\|'_{\mathfrak{A}_n(E,F)} = \sup\{|||P \circ R|||_{\mathfrak{A}_n(E_1,F)} : \ E_1 \text{ Banach space}, \|R\|_{\mathcal{L}(E_1,E)} = 1\}.$$

Then $\|\cdot\|_{\mathfrak{A}_n(E,F)} = \frac{\|\cdot\|'_{\mathfrak{A}_n(E,F)}}{\kappa_n}$ is an ideal norm on $\mathfrak{A}_n$ equivalent to $|||\cdot|||_{\mathfrak{A}_n}$. Moreover, using Corollary 2.1.7 we see that $1 \leq \kappa_n \leq e$. Now it is not difficult to prove the main result.

Theorem 3.1.18. *Let $\mathfrak{A}_{n+1}$ be a Banach ideal of $(n+1)$-homogeneous polynomials. Then there exists polynomial ideals $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n, \mathfrak{A}_{n+1}\}$ is a coherent sequence. The polynomial ideals $\mathfrak{A}_k$ are uniquely determined by $\mathfrak{A}_{n+1}$ and they can be normed to obtain constants of coherence $1 \leq C, D \leq e$.*

To finish this subsection we mention that analogously to Proposition 2.4.3 it can easily be proven that interpolation of coherent sequences is again coherent:

Proposition 3.1.19. *Let $\{\mathfrak{A}_k^0\}_k$ and $\{\mathfrak{A}_k^1\}_k$ be coherent sequences of polynomial ideals with constants C_0, D_0 and C_1, D_1, respectively. Let F is an exact interpolation functor of exponent θ. Then, the polynomial ideal $F(\mathfrak{A}_n)$ is compatible with the operator ideal $F(\mathfrak{A})$ with constants $C_0^{1-\theta}C_1^{\theta}$ and $D_0^{1-\theta}D_1^{\theta}$.*

3.1.1 The smallest and greatest coherent sequence associated to an operator ideal

In this subsection we show that given an operator ideal $\mathfrak{A}$, the ideals $\mathcal{F}_k^{\mathfrak{A}}$ and $\mathcal{M}_k^{\mathfrak{A}}$ defined in Section 2.2 are the smallest and the greatest sequences of polynomial ideals coherent with $\mathfrak{A}$.

Let $\mathfrak{A}$ be a linear operator ideal. If $\{\mathfrak{A}_k\}_k$ is any coherent sequence of normed ideals of homogeneous polynomials with $\mathfrak{A}_1 = \mathfrak{A}$, then for each $k \in \mathbb{N}$, $\mathfrak{A}_k$ is compatible with $\mathfrak{A}$. Thus, by Proposition 2.2.2 we have

$$\mathcal{F}_k^{\mathfrak{A}}(E,F) \subset \mathfrak{A}_k(E,F) \subset \mathcal{M}_k^{\mathfrak{A}}(E,F),$$

for every Banach spaces E and F. Therefore, if we show that $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ and $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ are coherent sequences (note that $\mathcal{M}_1^{\mathfrak{A}} = \mathcal{F}_1^{\mathfrak{A}} = \mathfrak{A}$), we can conclude that they are, respectively, the largest and the smallest coherent sequence associated to $\mathfrak{A}$. Analogously, if $\mathfrak{A}$ is complete, we obtain that $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence of Banach polynomial ideals associated to $\mathfrak{A}$. However, in the three cases the coherence constants obtained are larger than the compatibility constants of Proposition 2.2.2.

Proposition 3.1.20. *Let $\mathfrak{A}$ be a normed operator ideal. Then:*

(a) *The sequence $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = e$ and $D = 1$. Thus, $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ is the largest coherent sequence associated to $\mathfrak{A}$.*

(b) *The sequence $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = 1$ and $D = e$. Thus, $\{\mathcal{F}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence associated to $\mathfrak{A}$.*

(c) *If $\mathfrak{A}$ is complete, the sequence $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is coherent with constants $C = 1$ and $D = e$. Thus, $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$ is the smallest coherent sequence of Banach ideals associated to $\mathfrak{A}$.*

Proof. We prove conditions (i) and (ii) of Definition 3.1.1 for the sequences $\{\mathcal{M}_k^{\mathfrak{A}}\}_k$ and $\{\mathcal{N}_k^{\mathfrak{A}}\}_k$. The case (b) follows similarly.

(a) (i) Let $P \in \mathcal{M}_k^{\mathfrak{A}}(E,F)$ and $a \in E$. We have to show that $P_a \in \mathcal{M}_{k-1}^{\mathfrak{A}}(E,F)$ with $\|P_a\|_{\mathcal{M}_{k-1}^{\mathfrak{A}}(E,F)} \leq e\|a\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}$. For this, we need to prove that $(P_a)_{b^{k-2}} \in \mathfrak{A}(E,F)$ for all $b \in S_E$ and $\|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E,F)} \leq e\|a\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}$.

As in Lemma 2.1.6 we take $r \in \mathbb{C}$ a primitive $(k-1)$-root of 1. Then, for each $x \in E$ and $t > 0$, we have

$$\begin{aligned}
(P_a)_{b^{k-2}}(x) &= (P_x)^\vee(a, b^{k-2}) \\
&= \frac{1}{(k-1)^2}\sum_{j=0}^{k-2} r^j t^{k-2} P_x\left(\frac{r^j}{t}b + a\right) \\
&= \frac{1}{(k-1)^2}\sum_{j=0}^{k-2} r^j t^{k-2} P_{(\frac{r^j}{t}b+a)^{k-1}}(x).
\end{aligned}$$

Since $P \in \mathcal{M}_k^{\mathfrak{A}}(E,F)$, $(P_a)_{b^{k-2}}$ belongs to $\mathfrak{A}(E,F)$.

Choosing $t = \frac{1}{k-2}$ we obtain, for $\|a\| = \|b\| = 1$, that

$$\begin{aligned}
\|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E,F)} &\leq \tfrac{1}{(k-1)^2}\sum_{j=0}^{k-2} t^{k-2}\left\|P_{(\frac{r^j}{t}b+a)^{k-1}}\right\|_{\mathfrak{A}(E,F)} \\
&\leq \tfrac{1}{(k-1)^2}\sum_{j=0}^{k-2}\left(\frac{1}{k-2}\right)^{k-2}(k-1)^{k-1}\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)} \\
&\leq e\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}.
\end{aligned}$$

Therefore, for each $a \in E$,

$$\|P_a\|_{\mathcal{M}_{k-1}^{\mathfrak{A}}(E,F)} = \sup_{\|b\|=1}\|(P_a)_{b^{k-2}}\|_{\mathfrak{A}(E,F)} \leq e\|a\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}.$$

(ii) Let $P \in \mathcal{M}_k^{\mathfrak{A}}(E,F)$, $\gamma \in E'$ and $a \in E$. Then

$$(\gamma P)_{a^k} = \tfrac{1}{k+1}\Big(P(a)\gamma + k\gamma(a)P_{a^{k-1}}\Big).$$

This implies that $(\gamma P)_{a^k} \in \mathfrak{A}(E,F)$ and thus $\gamma P \in \mathcal{M}_{k+1}^{\mathfrak{A}}(E,F)$. Moreover,

$$\begin{aligned}
\|\gamma P\|_{\mathcal{M}_{k+1}^{\mathfrak{A}}(E,F)} &= \sup_{\|a\|=1}\|(\gamma P)_{a^k}\|_{\mathfrak{A}(E,F)} \\
&\leq \frac{1}{k+1}\sup_{\|a\|=1}\Big(\|P\|_{\mathcal{P}^k(E,F)}\|\gamma\| + k\|\gamma\|\|P_{a^{k-1}}\|_{\mathfrak{A}(E,F)}\Big) \\
&\leq \|\gamma\|\|P\|_{\mathcal{M}_k^{\mathfrak{A}}(E,F)}.
\end{aligned}$$

(c) (i) For $P \in \mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)$, fix a representation $P = \sum_{i=1}^\infty \gamma_i^k T_i$ and let $a \in E$. Then

$$P_a = \frac{1}{k+1}\sum_{i=1}^\infty\Big(T_i(a)\gamma_i^k + k\gamma_i(a)(\gamma_i^{k-1}T_i)\Big).$$

The partial sums of the above series belong to $\mathcal{N}_k^{\mathfrak{A}}(E,F)$. Furthermore,

$$\begin{aligned}\frac{1}{k+1}\sum_{i=1}^{\infty}\Big\|\Big(T_i(a)\gamma_i^k + k\gamma_i(a)(\gamma_i^{k-1}T_i)\Big)\Big\|_{\mathcal{N}_k^{\mathfrak{A}}(E,F)} &\\ \leq \sum_{i=1}^{\infty}\frac{\|a\|\|T_i\|_{\mathcal{L}(E,F)}\|\gamma_i\|^k + k\|a\|\|\gamma_i\|^k\|T_i\|_{\mathfrak{A}(E,F)}}{k+1} &\\ \leq \|a\|\sum_{i=1}^{\infty}\|\gamma_i\|^k\|T_i\|_{\mathfrak{A}(E,F)}. &\end{aligned}$$

Then, P_a belongs to $\mathcal{N}_k^{\mathfrak{A}}(E,F)$ and, since the above inequality is valid for every representation of P, we obtain that

$$\|P_a\|_{\mathcal{N}_k^{\mathfrak{A}}(E,F)} \leq \|a\|\|P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)}.$$

(ii) Let $P\in\mathcal{N}_k^{\mathfrak{A}}(E,F)$. Suppose first that $P=\gamma^{k-1}T$, with $\gamma\in E'$ and $T\in\mathcal{L}(E,F)$. Let $\phi\in E'$. Then, proceeding as in Lemma 2.1.6 and Corollary 2.1.7 we obtain the expression

$$(\phi P)(x) = (\phi\gamma^{k-1}T)(x) = \frac{1}{k^2}\sum_{j=0}^{k-1}t^{k-1}r^j\left(\frac{r^j}{t}\gamma(x)+\phi(x)\right)^k T(x),$$

where $t>0$ and $r\in\mathbb{C}$ is a primary k-root of the unit. This means that $\phi P\in\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)$ and

$$\|\phi P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)} \leq e\|\phi\|\|\gamma\|^{k-1}\|T\|_{\mathfrak{A}(E,F)}.$$

Consider now $P\in\mathcal{N}_k^{\mathfrak{A}}(E,F)$ and a representation $P=\sum_{i=1}^{\infty}\gamma_i^{k-1}T_i$ with $\gamma_i\in E'$ and $T_i\in\mathcal{L}(E,F)$. Then the finite sums $\sum_{i=1}^{m}\phi\gamma_i^{k-1}T_i$ belong to $\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)$ and the series converges since

$$\sum_{i=1}^{\infty}\|\phi\gamma_i^{k-1}T_i\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)} \leq e\|\phi\|\sum_{i=1}^{\infty}\|\gamma_i\|^{k-1}\|T_i\|_{\mathfrak{A}(E,F)}.$$

Thus $P\in\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)$. The above inequality is valid for every representation of P, hence

$$\|\phi P\|_{\mathcal{N}_{k+1}^{\mathfrak{A}}(E,F)} \leq e\|\phi\|\|P\|_{\mathcal{N}_k^{\mathfrak{A}}(E,F)}.$$

$\square$

3.1.2 Composition ideals

This subsection is devoted to show that the composition of a coherent sequence of polynomial ideals with two fixed operator ideals is still coherent in the following situations: if either both linear operator ideals are closed or the polynomial ideal is closed. For the first case we have the following:

Proposition 3.1.21. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of normed polynomial ideals with constants C and D, and let $\mathfrak{C}$ and $\mathfrak{B}$ be closed ideals of linear operators. Then $\{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}\}_k$ is a coherent sequence with constants C and D.*

Proof. We check condition (i) of Definition 2.1.2: Let $P \in \mathfrak{C} \circ \mathfrak{A}_k \circ \mathfrak{B}(E, F)$. Then $P = SQT$ with $S \in \mathfrak{C}(\widetilde{F}, F)$, $T \in \mathfrak{B}(E, \widetilde{E})$ y $Q \in \mathfrak{A}_k(\widetilde{E}, \widetilde{F})$.

Also, $P_a(\cdot) = S\overset{\vee}{Q}(T(a), T(\cdot), \dots, T(\cdot)) = SQ_{T(a)}T$, and $Q_{T(a)} \in \mathfrak{A}_{k-1}(\widetilde{E}, \widetilde{F})$ since $\{\mathfrak{A}_k\}$ is coherent. Therefore $P_a \in \mathfrak{C} \circ \mathfrak{A}_{k-1} \circ \mathfrak{B}(E, F)$.

Moreover,

$$\begin{aligned}
\|P_a\|_{\mathfrak{C}\circ\mathfrak{A}_{k-1}\circ\mathfrak{B}(E,F)} &\leq C\|T(a)\|\|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\|T\|^{k-1}_{\mathfrak{B}(E,\widetilde{E})} \\
&\leq C\|a\|\|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\|T\|^{k}_{\mathfrak{B}(E,\widetilde{E})},
\end{aligned}$$

and this holds for every factorization $P = SQT$. Hence

$$\|P_a\|_{\mathfrak{C}\circ\mathfrak{A}_{k-1}\circ\mathfrak{B}(E,F)} \leq C\|a\|\|P\|_{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}(E,F)}.$$

Condition (ii): Again, we take $P = SQT$ with $S \in \mathfrak{C}(\widetilde{F}, F)$, $T \in \mathfrak{B}(E, \widetilde{E})$ and $Q \in \mathfrak{A}_k(\widetilde{E}, \widetilde{F})$.

Consider $\gamma \in E'$ and define the operators

$$\begin{aligned}
&\widetilde{T} \in \mathfrak{B}(E, \widetilde{E} \times \mathbb{C}), && \widetilde{T}(x) = (T(x), \gamma(x)) = (i_1 \circ T)(x) + (i_2 \circ \gamma)(x) \\
&R \in \mathfrak{A}_{k+1}(\widetilde{E} \times \mathbb{C}, \widetilde{F}), && R(y, \lambda) = Q(y)\lambda = (Q \circ \pi_1)(y, \lambda) \cdot \pi_2(y, \lambda),
\end{aligned}$$

where i_1, i_2 are the inclusions

$$\widetilde{E} \overset{i_1}{\hookrightarrow} \widetilde{E} \times \mathbb{C}, \qquad \mathbb{C} \overset{i_2}{\hookrightarrow} \widetilde{E} \times \mathbb{C},$$

and π_1, π_2 are the projections to the first and second coordinate.

We have that $SR\widetilde{T}$ belongs to $\mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$ and

$$SR\widetilde{T}(x) = SR(T(x), \gamma(x)) = S(\gamma(x)Q(T(x))) = \gamma(x)P(x).$$

As a consequence, $\gamma P \in \mathfrak{C} \circ \mathfrak{A}_{k+1} \circ \mathfrak{B}(E, F)$.

For the inequality of norms, we may assume that $\|T\|_{\mathfrak{B}(E,\widetilde{E})} = \|T\|_{\mathfrak{L}(E,\widetilde{E})} = 1$. Let us consider in $\widetilde{E} \times \mathbb{C}$ the norm $\|(y, \lambda)\| = \max\{\|y\|, |\lambda|\}$ and suppose $\|\gamma\| = 1$. Then, we have

$$\|R\|_{\mathfrak{A}_{k+1}(\widetilde{E}\times\mathbb{C},\widetilde{F})} \leq D\|Q \circ \pi_1\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\|\pi_2\| \leq D\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})},$$

and

$$\|\widetilde{T}\|_{\mathfrak{B}(E,\widetilde{E}\times\mathbb{C})} = \|\widetilde{T}\|_{\mathcal{L}(E,\widetilde{E}\times\mathbb{C})} = \max\{\|T\|_{\mathfrak{L}(E,\widetilde{E})}, \|\gamma\|\} = 1.$$

Thus we obtain

$$\begin{aligned}
\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)} &\leq \|S\|_{\mathfrak{C}(\widetilde{F},F)}\|R\|_{\mathfrak{A}_{k+1}(\widetilde{E}\times\mathbb{C},\widetilde{F})}\|\widetilde{T}\|^{k+1}_{\mathfrak{B}(E,\widetilde{E}\times\mathbb{C})} \\
&\leq D\|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}
\end{aligned}$$

and this is true for every factorization $P = SQT$ with $\|T\|_{\mathfrak{B}(E,\widetilde{E})} = 1$. Hence, for a general $\gamma \in E'$,

$$\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)} \leq D\|\gamma\|\|P\|_{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}(E,F)}.$$

□

For the second case we can prove:

Proposition 3.1.22. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of closed polynomial ideals. If $\mathfrak{B}$ and $\mathfrak{C}$ are normed operator ideals, then $\{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}\}_k$ is a coherent sequence with constants $C=e$ and $D=1$.*

Proof. Condition (i): Let $P=SQT$, with $T\in\mathfrak{B}(E,\widetilde{E})$, $Q\in\mathfrak{A}_k(\widetilde{E},\widetilde{F})$ and $S\in\mathfrak{C}(\widetilde{F},F)$. Then $P_a=(SQT)_a=SQ_{T(a)}T\in\mathfrak{C}\circ\mathfrak{A}_{k-1}\circ\mathfrak{B}(E,F)$.

Moreover, by Example 2.1.8,

$$\begin{aligned}\|P_a\|_{\mathfrak{C}\circ\mathfrak{A}_{k-1}\circ\mathfrak{B}(E,F)} &\leq \|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q_{T(a)}\|_{\mathfrak{A}_{k-1}(\widetilde{E},\widetilde{F})}\|T\|^{k-1}_{\mathfrak{B}(E,\widetilde{E})}\\ &\leq e\|a\|\|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\|T\|^{k}_{\mathfrak{B}(E,\widetilde{E})}.\end{aligned}$$

Since this holds for every factorization $P=SQT$, we obtain $\|P_a\|_{\mathfrak{C}\circ\mathfrak{A}_{k-1}\circ\mathfrak{B}(E,F)}\leq e\|a\|\|P\|_{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}(E,F)}$.

Condition (ii): Let $P=SQT$, $T\in\mathfrak{B}(E,\widetilde{E})$, $Q\in\mathfrak{A}_k(\widetilde{E},\widetilde{F})$ and $S\in\mathfrak{C}(\widetilde{F},F)$. Define as in Proposition 3.1.21 the operators

$$\begin{aligned}\widetilde{T}&\in\mathfrak{B}(E,\widetilde{E}\times\mathbb{C}), &\widetilde{T}(x)&=(T(x),\gamma(x))=(i_1\circ T)(x)+(i_2\circ\gamma)(x)\\ R&\in\mathfrak{A}_{k+1}(\widetilde{E}\times\mathbb{C},\widetilde{F}), &R(y,\lambda)&=Q(y)\lambda=(Q\circ\pi_1)(y,\lambda)\cdot\pi_2(y,\lambda).\end{aligned}$$

Then, $SR\widetilde{T}\in\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)$ and

$$SR\widetilde{T}(x)=S(R(T(x),\gamma(x)))=S(Q(T(x)))\gamma(x)=\gamma(x)P(x).$$

Thus $\gamma P\in\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)$.

To prove the inequality of norms we now consider $\widetilde{E}\times\mathbb{C}$ with the norm $\|(y,\lambda)\|=\|y\|+|\lambda|$. Since, for every k, the norm in $\mathfrak{A}_k$ is the usual polynomial norm,

$$\begin{aligned}\|R\|_{\mathfrak{A}_{k+1}(\widetilde{E}\times\mathbb{C},\widetilde{F})} &= \sup_{\|y\|+|\lambda|\leq1}\|\lambda Q(y)\|\leq\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\sup_{\|y\|+|\lambda|\leq1}|\lambda|\|y\|^k\\ &\leq \frac{k^k}{(k+1)^{k+1}}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}.\end{aligned}$$

Also,

$$\begin{aligned}\|\widetilde{T}\|_{\mathfrak{B}(E,\widetilde{E}\times\mathbb{C})} &= \|(i_1\circ T)(\cdot)+(i_2\circ\gamma)(\cdot)\|_{\mathfrak{B}(E,\widetilde{E}\times\mathbb{C})}\leq\|i_1\|\|T\|_{\mathfrak{B}(E,\widetilde{E})}+\|i_2\|\|\gamma\|\\ &= \|T\|_{\mathfrak{B}(E,\widetilde{E})}+\|\gamma\|.\end{aligned}$$

Then,

$$\begin{aligned}\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)} &\leq \|S\|_{\mathfrak{C}(\widetilde{F},F)}\|R\|_{\mathfrak{A}_{k+1}(\widetilde{E}\times\mathbb{C},\widetilde{F})}\|\widetilde{T}\|^{k+1}_{\mathfrak{B}(E,\widetilde{E}\times\mathbb{C})}\\ &\leq \|S\|_{\mathfrak{C}(\widetilde{F},F)}\frac{k^k}{(k+1)^{k+1}}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\Big(\|T\|_{\mathfrak{B}(E,\widetilde{E})}+\|\gamma\|\Big)^{k+1}.\end{aligned}$$

If we consider $T_t=tT$ and $Q_t=t^{-k}Q$, we obtain a new factorization of P and thus of γP. The previous inequality applied to this factorization gives

$$\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)}\leq\|S\|_{\mathfrak{C}(\widetilde{F},F)}\frac{k^k}{(k+1)^{k+1}}\frac{\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}}{t^k}\Big(t\|T\|_{\mathfrak{B}(E,\widetilde{E})}+\|\gamma\|\Big)^{k+1}.$$

This expression is minimum when $t = \frac{k\|\gamma\|}{\|T\|_{\mathfrak{B}(E,\widetilde{E})}}$. In this case,

$$\begin{aligned}
&\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)}\\
&\quad\leq \|S\|_{\mathfrak{C}(\widetilde{F},F)}\frac{k^k}{(k+1)^{k+1}}\Big(\frac{\|T\|_{\mathfrak{B}(E,\widetilde{E})}}{k\|\gamma\|}\Big)^k\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\big((k+1)\|\gamma\|\big)^{k+1}\\
&\quad= \|\gamma\|\|S\|_{\mathfrak{C}(\widetilde{F},F)}\|Q\|_{\mathfrak{A}_k(\widetilde{E},\widetilde{F})}\|T\|^k_{\mathfrak{B}(E,\widetilde{E})}.
\end{aligned}$$

This is true for each factorization $P = SQT$ and therefore,

$$\|\gamma P\|_{\mathfrak{C}\circ\mathfrak{A}_{k+1}\circ\mathfrak{B}(E,F)} \leq \|\gamma\|\|P\|_{\mathfrak{C}\circ\mathfrak{A}_k\circ\mathfrak{B}(E,F)}, \tag{3.2}$$

which completes the proof. □

As a consequence we have the following coherent sequences:

Example 3.1.23. 1. Let $1 < r < \infty$ and N be the largest integer not greater than r. Then the sequence of quasi-normed ideals of polynomials $\{\Pi_r, \mathcal{D}_r^2, \mathcal{D}_r^3, \dots\}$ (the ideals $\mathcal{D}_r^n$ are normed for $n \leq r$) which is coherent with constants $C = e$ and $D = 1$. Indeed, it was proved in [Sch91] that $\mathcal{P}^n \circ \Pi_r$.

2. The sequence r-factorable (strongly r-factorable, r-compact) polynomials is coherent since it is the composition of all the polynomials with the r-factorable (strongly r-factorable, r-compact) operators [Hol86].

3.1.3 Relation with tensor norms

Suppose that for each k we have a symmetric k-tensor norm α_k and set $\mathfrak{A}_k(E,F) = \mathfrak{A}(\bigotimes^{k,s}_{\alpha_k} E, F)$. By Proposition 2.5.1 each $\mathfrak{A}_k$ is compatible with $\mathfrak{A}$ but, in order to obtain a coherent sequence, some coherence properties for the sequence of tensor norms $\{\alpha_k\}_k$ are needed. Let us establish this coherence. For $a \in E$, $\gamma \in E'$ we define the following mappings for each k (we omit the dependence on k in the notation for the sake of simplicity):

$$\begin{array}{rcl} \Phi_a : \bigotimes^{k-1,s} E & \rightarrow & \bigotimes^{k,s} E \\ x^{k-1} & \mapsto & \sigma(a \otimes x^{k-1}) \end{array} \qquad \begin{array}{rcl} \Psi_\gamma : \bigotimes^{k+1,s} E & \rightarrow & \bigotimes^{k,s} E \\ x^{k+1} & \mapsto & \gamma(x)x^k. \end{array}$$

Proposition 3.1.24. *Let $\mathfrak{A}$ be an operator ideal. Suppose we have, for each k, a symmetric k-tensor norm α_k and define $\mathfrak{A}_k(E,F) = \mathfrak{A}(\bigotimes^{k,s}_{\alpha_k} E, F)$. Then, $\{\mathfrak{A}_k\}_k$ is a coherent sequence of polynomial ideals (with constants C and D) if and only if for every Banach space E the mappings*

$$\Phi_a : \Big(\bigotimes^{k-1,s} E, \alpha_{k-1}\Big) \longrightarrow \Big(\bigotimes^{k,s} E, \alpha_k\Big)$$

and

$$\Psi_\gamma : \Big(\bigotimes^{k+1,s} E, \alpha_{k+1}\Big) \longrightarrow \Big(\bigotimes^{k,s} E, \alpha_k\Big)$$

are continuous for every k (with $\|\Phi_a\| \leq C\|a\|$ and $\|\Psi_\gamma\| \leq D\|\gamma\|$).

Proof. Just note that if $P \in \mathcal{P}^k(E)$, we have $T_{P_a} = T_P \circ \Phi_a$ and $T_{\gamma P} = T_P \circ \Psi_\gamma$. □

Corollary 3.1.25. *The sequences $\{\mathcal{P}^k_{PI}\}_k$ and $\{\mathcal{P}^k_{GI}\}_k$ of Piestch and Grothendieck integral polynomials are coherent with constants $C = 1$ and $D = e$. Moreover, they are compatible with the ideals of Piestch and Grothendieck integral operators respectively, with constants $A = 1$ and $B = e$.*

Proof. Recall that $\mathcal{P}^k_{PI}(E, F) = \mathcal{L}_{PI}(\bigotimes^{k,s}_{\varepsilon_s} E, F)$ and $\mathcal{P}^k_{GI}(E, F) = \mathcal{L}_{GI}(\bigotimes^{k,s}_{\varepsilon_s} E, F)$ isometrically [CL05, Vil03].

As in Corollary 2.5.2, a direct computation shows that the application Φ_a from the previous proposition is continuous with $C = 1$.

Let $z = \sum x_i^{k+1} \in \bigotimes^{k+1,s} E$, and define $Q^z \in \mathcal{P}^{k+1}(E, F)$ as $Q^z(\phi) = \sum \phi(x_i)^{k+1}$. Then $\|Q^z\| = \varepsilon_s^{k+1}(z)$. Moreover, by Corollary 2.1.7(*b*), $\sum \gamma(x_i)\phi(x_i)^k = |\overset{\vee}{Q^z}(\gamma, \phi^k)| \leq e\|Q^z\|\|\gamma\|\|\phi\|^k$. Since this is true for every $\phi \in E'$, taking supremum with $\phi \in B_{E'}$, we have that

$$\varepsilon_s^k(\Psi_\gamma(z)) = \varepsilon_s^k\left(\sum \gamma(x_i)x_i^k\right) \leq e\|\gamma\|\varepsilon_s^{k+1}(z).$$

□

In Section 2.5 we showed that there are mixed tensor norms that are not equivalent to any (α, β)-norm (Corollary 2.5.7). We give now a criterion of coherence for ideals which are dual to mixed tensor norms. For $a \in E$ and $\gamma \in E'$ we define the following mappings:

$$\begin{array}{rcl} \Phi^F_a : \bigotimes^{k-1,s} E \otimes F & \to & \bigotimes^{k,s} E \otimes F \\ x^{k-1} \otimes y & \mapsto & \sigma(a \otimes x^{k-1}) \otimes y \end{array} \qquad \begin{array}{rcl} \Psi^F_\gamma : \bigotimes^{k+1,s} E \otimes F & \to & \bigotimes^{k,s} E \otimes F \\ x^{k+1} \otimes y & \mapsto & \gamma(x)x^k \otimes y \end{array}$$

Proposition 3.1.26. *Let $\mathfrak{A}$ be an operator ideal. Suppose we have, for each k, a mixed tensor norm δ_k of order $k+1$. Then, $\mathcal{P}^k_{\delta_k}$ is a coherent sequence of polynomial ideals (with constants C and D) if and only if the mappings $\Phi^{F'}_a$ and $\Psi^{F'}_\gamma$ are δ_{k-1}-to-δ_k and δ_{k+1}-to-δ_k continuous for every k, E and F (with $\|\Phi^{F'}_a\| \leq C\|a\|$ and $\|\Psi^{F'}_\gamma\| \leq D\|\gamma\|$).*

Proof. Just note that conditions (i) and (ii) of Definition 3.1.1 are dual to continuity properties of the mappings defined above when we consider F' instead of F. □

We finish this subsection proving that if we have a finitely generated k-fold symmetric tensor norm α_k then the sequence of maximal (scalar) ideals associated to α_k is coherent if and only if the sequence of minimal ideals associated to α_k is coherent. We will use the Representation Theorem for maximal polynomial ideals [FH02, Theorem 3.2]. First we need the following lemmata, which will be useful in Subsection 3.1.5.

Lemma 3.1.27. *For each k, let α_k be a finitely generated k-fold symmetric tensor norm α_k. Consider $\mathfrak{A}^{max}_k$ and $\mathfrak{A}^{min}_k$ maximal and minimal ideals associated to α_k. Then the following are equivalent.*

(i) For every Banach space E, if $P \in \mathfrak{A}^{max}_k(E)$ and $a \in E$ then $P_a \in \mathfrak{A}^{max}_{k-1}(E)$ and

$$\|P_a\|_{\mathfrak{A}^{max}_{k-1}(E)} \leq c_k\|P\|_{\mathfrak{A}^{max}_k(E)}\|a\|.$$

(ii) For every Banach space E, if $P \in \mathfrak{A}^{min}_k(E)$ and $a \in E$ then $P_a \in \mathfrak{A}^{min}_{k-1}(E)$ and

$$\|P_a\|_{\mathfrak{A}^{min}_{k-1}(E)} \leq c_k\|P\|_{\mathfrak{A}^{min}_k(E)}\|a\|.$$

(iii) For every Banach space E, if $s = \sum_{j=1}^n \gamma_j^k$ is a tensor in $\bigotimes^{k,s}_{\alpha_k} E'$ and $a \in E$, then

$$\alpha_{k-1}(\sum_j \gamma(a)\gamma_j^{k-1}) \leq c_k\alpha_k(s)\|a\|.$$

Proof. The three statements are equivalent if E is a finite dimensional Banach space because for finite dimensional spaces $\mathfrak{A}_k^{max}(E) \overset{1}{=} \mathfrak{A}_k^{min}(E) \overset{1}{=} \bigotimes_{\alpha_k}^{k,s} E'$. Since α_k is finitely generated, (i) is implied by (ii) or by (iii) for all Banach spaces.

We now prove that (i) implies (iii). Note that (iii) is equivalent to prove that the bilinear map $\psi_E : \left(E \times \bigotimes_{\alpha_k}^{k-1,s} E', \|\cdot\|_\infty\right) \to \bigotimes_{\alpha_{k-1}}^{k-1,s} E'$, $\psi_E(a, \sum \gamma_j^k) = \sum \gamma(a)\gamma_j^{k-1}$ is continuous of norm $\leq c_k$ for every Banach space E. If (i) is true then ψ_S is continuous (with norm $\leq c_k$) for every finite dimensional Banach space S. Let M be a finite dimensional subspace of E' such that $\sum \gamma_j^k \in \bigotimes^{k,s} M$ and denote by $[a]$ the subspace of E generated by a. Then

$$\begin{aligned} \alpha_{k-1}\big(\sum \gamma(a)\gamma_j^{k-1}, \bigotimes^{k-1,s} M + [a]\big) &\leq c_k \max\{\alpha_k\big(\sum \gamma_j^k, \bigotimes^{k,s} M + [a]\big), \|a\|\}\} \\ &\leq c_k \max\{\alpha_k\big(\sum \gamma_j^k, \bigotimes^{k,s} M\big), \|a\|\}\}, \end{aligned}$$

where the second inequality is true by the metric mapping property. Taking the infimum over M we obtain that $\|\psi_E\| \leq c_k$ and thus we have (iii).

To see that (i) implies (ii), just note that $\mathfrak{A}_k^{min} = \mathfrak{A}_k^{max} \circ \overline{\mathcal{F}}$ and use Proposition 3.1.21 (proof of condition i). □

The following lemma may be proved similarly, moreover it can also be seen as a particular case of Proposition 4.1.12.

Lemma 3.1.28. *For each k, let α_k be a finitely generated k-fold symmetric tensor norm α_k. Consider $\mathfrak{A}_k^{max}$ and $\mathfrak{A}_k^{min}$ maximal and minimal ideals associated to α_k. Then the following are equivalent.*
(i) For every Banach space E, if $P \in \mathfrak{A}_k^{max}(E)$ and $\gamma \in E'$ then $\gamma P \in \mathfrak{A}_{k+1}^{max}(E)$ and

$$\|\gamma P\|_{\mathfrak{A}_{k+l}^{max}(E)} \leq c_k \|P\|_{\mathfrak{A}_k^{max}(E)} \|\gamma\|.$$

(ii) For every Banach space E, if $P \in \mathfrak{A}_k^{min}(E)$ and $\gamma \in E'$ then $\gamma P \in \mathfrak{A}_{k+1}^{min}(E)$ and

$$\|\gamma P\|_{\mathfrak{A}_{k+l}^{min}(E)} \leq c_k \|P\|_{\mathfrak{A}_k^{min}(E)} \|\gamma\|.$$

(iii) For every Banach space E, if $s \in \bigotimes_{\alpha_k}^{k,s} E'$ and $\gamma \in E'$, then

$$\alpha_{k+1}(\sigma(s \otimes \gamma)) \leq c_k \alpha_k(s) \|\gamma\|.$$

We have thus the following corollary for sequences of scalar polynomial ideals:

Corollary 3.1.29. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a sequence of scalar polynomials ideals, then $\{\mathfrak{A}_k^{max}\}_k$ is a coherent sequence if and only if $\{\mathfrak{A}_k^{min}\}_k$ is coherent.*

If moreover $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ is coherent then $\{\mathfrak{A}_k^{min}\}_k$ and $\{\mathfrak{A}_k^{max}\}_k$ are coherent sequences.

Maximal and minimal hulls will be revisited in the next subsection and the vector valued case will be treated there.

3.1.4 Maximal, minimal and adjoint ideals

In this subsection we study the stability of the coherence conditions under the operation of taking adjoints, maximal and minimal hulls of the polynomial ideals.

If a sequence of polynomial ideals form a holomorphy type, then the sequence of adjoint ideals need not to be a holomorphy type. On the other hand, the two operations involved in the definition of coherent sequence (fixing variables and multiplication by linear functionals) are dual operations. This allows us to prove the following.

Proposition 3.1.30. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence with constants C and D. Then $\{\mathfrak{A}_k^*\}_k$ is a coherent sequence with constants D and C.*

Proof. Condition (i). By Proposition 3.1.26 it suffices to verify that

$$\begin{array}{rcl} \Phi_a^{F'}: \Big(\bigotimes^{k-1,s} E\otimes F', \lambda_{k-1}\Big) & \to & \Big(\bigotimes^{k,s} E\otimes F', \lambda_k\Big) \\ x^{k-1}\otimes y & \mapsto & \sigma(a\otimes x^{k-1})\otimes y \end{array}$$

is continuous and that $\|\Phi_a^{F'}\| \le D\|a\|$, for all $a\in E$; where the λ_k's are the tensor norms predual to the ideals $\mathfrak{A}_k^*$ (see Preliminaries 1.2.1) .

Consider $M \in FIN(E)$; $N\in FIN(F')$ and $z=\sum_i x_i^{k-1}\otimes y_i \in \bigotimes^{k-1,s} M\otimes N$. Then $P^{\Phi_a^{F'}(z)} \in \mathfrak{A}_k(M_a', N)$, where $M_a = M\bigoplus[a]$. If $x'\in M_a'$

$$P^{\Phi_a^{F'}(z)}(x') = \sum_i \sigma(a\otimes x_i^{k-1})(x')y_i = \sum_i x'(a)x'(x_i)^{k-1}y_i = (aP^z)(x').$$

Thus,

$$\begin{array}{rcl} \lambda_k\Big(\Phi_a^{F'}(z); \bigotimes^{k,s} E\otimes F'\Big) & \le & \lambda_k\Big(\Phi_a^{F'}(z); \bigotimes^{k,s} M_a\otimes N\Big) \\ & = & \|P^{\Phi_a^{F'}(z)}\|_{\mathfrak{A}_k(M_a',N)} = \|aP^z\|_{\mathfrak{A}_k(M_a',N)} \\ & \le & D\|a\| \|P^z\|_{\mathfrak{A}_{k-1}(M_a',N)} \\ & \le & D\|a\| \ \lambda_{k-1}\big(z; \bigotimes^{k-1,s} M\otimes N\big). \end{array}$$

And hence, $\lambda_k\Big(\Phi_a^{F'}(z); \bigotimes^{k,s} E\otimes F'\Big) \le D\|a\|\lambda_{k-1}\big(z;\bigotimes^{k-1,s} E\otimes F'\big)$.

Condition (ii). We must verify that the mapping

$$\begin{array}{rcl} \Psi_\gamma^{F'}: \Big(\bigotimes^{k+1,s} E\otimes F', \lambda_{k+1}\Big) & \to & \Big(\bigotimes^{k,s} E\otimes F', \lambda_k\Big) \\ x^{k+1}\otimes y & \mapsto & \gamma(x)x^k\otimes y \end{array}$$

is continuous and that $\|\Psi_\gamma^{F'}\| \le C\|\gamma\|$, for all $\gamma\in E'$.

Consider $\gamma \in E'$ and let $M\in FIN(E)$, $N\in FIN(F')$ and $z_i = \sum x_i^{k+1}\otimes y_i \in \bigotimes^{k+1,s} M\otimes N$. Then $P^{\Psi_\gamma^{F'}(z)} \in \mathfrak{A}_k(M', N)$ and, if $x'\in M'$,

$$P^{\Psi_\gamma^{F'}(z)}(x') = \sum_i \gamma(x_i)x_i(x')^k y_i = \sum_i x_i(\gamma)x_i(x')^k y_i = (P^z)_\gamma(x').$$

Therefore,

$$\begin{array}{rcl} \lambda_k\Big(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} E\otimes F'\Big) & \le & \lambda_k\Big(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} M\otimes N\Big) \\ & = & \big\|P^{\Psi_\gamma^{F'}(z)}\big\|_{\mathfrak{A}_k(M',N)} = \|(P^z)_\gamma\|_{\mathfrak{A}_k(M',N)} \\ & \le & C\|\gamma\| \|P^z\|_{\mathfrak{A}_{k+1}(M',N)} \\ & = & C\|\gamma\| \ \lambda_{k+1}\big(z; \bigotimes^{k+1,s} M\otimes N\big). \end{array}$$

Thus, $\lambda_k\Big(\Psi_\gamma^{F'}(z); \bigotimes^{k,s} E\otimes F'\Big) \leq C\|\gamma\|\lambda_{k+1}\big(z; \bigotimes^{k+1,s} E\otimes F'\big)$. □

Since $\mathfrak{A}_k^{\max}$ coincides with $\mathfrak{A}_k^{**}$, the sequence of maximal hulls of the polynomial ideals $\mathfrak{A}_k$ preserves the coherence of the original sequence:

Corollary 3.1.31. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence with constants C and D. Then $\{\mathfrak{A}_k^{\max}\}_k$ is a coherent sequence with constants C and D.*

The preservation of coherence under minimal hulls is a particular case of Proposition 3.1.21 since $\mathfrak{A}_n^{\min} = \overline{\mathcal{F}} \circ \mathfrak{A}_n^{\min} \circ \overline{\mathcal{F}}$.

Corollary 3.1.32. *Let $\{\mathfrak{A}_k\}_k$ be a coherent sequence of Banach polynomial ideals with constants C and D. Then $\{\mathfrak{A}_k^{\min}\}_k$ is a coherent sequence with constants C and D.*

Remark 3.1.33. The reciprocal of Proposition 3.1.30 and Corollaries 3.1.31 and 3.1.32 is not true. A counterexample for the three is the sequence $\{\mathfrak{A}_k\}_k$, where

$$\mathfrak{A}_k(E,F) = \begin{cases} \mathcal{P}_N^k(E,F) & \text{if } k \text{ is even} \\ \mathcal{P}_{GI}^k(E,F) & \text{if } k \text{ is odd.} \end{cases}$$

Then $\{\mathfrak{A}_k^{\max}\}_k = \{\mathcal{P}_{GI}^k\}$, $\{\mathfrak{A}_k^*\}_k = \{\mathcal{P}^k\}_k$ and $\{\mathfrak{A}_k^{\min}\}_k = \{\mathcal{P}_N^k\}_k$ are coherent sequences, but $\{\mathfrak{A}_k\}_k$ is not.

3.1.5 Sequences of polynomial ideals associated to natural tensor norms

In [CGb], natural tensor norms for arbitrary order are introduced and studied, in the spirit of the natural tensor norms of Grothendieck. We will prove that the sequences of polynomial ideals associated to the symmetric natural tensor norms are coherent. First we introduce some notation to recall their definition.

For a symmetric tensor norm α_k (of order k), the projective and injective associates (or hulls) of α_k will be denoted, by extrapolation of the 2-fold case, as $\backslash\alpha_k/$ and $/\alpha_k\backslash$ respectively. They are defined as the tensor norms induced by the following mappings:

$$\big(\otimes^{k,s}\ell_1(B_E),\alpha_k\big) \overset{1}{\twoheadrightarrow} \big(\otimes^{k,s} E, \backslash\alpha_k/\big).$$

$$\big(\otimes^{k,s} E, /\alpha_k\backslash\big) \overset{1}{\hookrightarrow} \big(\otimes^{k,s}\ell_\infty(B_{E'}),\alpha_k\big).$$

Recall that for a symmetric tensor norm of order k, α_k, its dual tensor norm α_k' is defined on finite dimensional normed spaces by

$$\big(\otimes^{k,s} M, \alpha_k'\big) :\overset{1}{=} [\big(\otimes^{k,s} M', \alpha_k\big)]'$$

and then extended to Banach spaces so that it is finitely generated.

We say that α_k is a natural symmetric tensor norm (of order k) if α_k is obtained from π_k with a finite number of the operations $\backslash\ /$, $/\ \backslash$, $'$.

For $k \geq 3$, it is shown in [CGb] that there are exactly six non-equivalent natural tensor norms (note that for $k = 2$ there are only four). They can be arranged in the following diagram:

$$\begin{array}{ccccc} & & \pi_k & & \\ & & \uparrow & & \\ & & \backslash/\pi_k\backslash/ & & \\ & \nearrow & & \nwarrow & \\ /\pi_k\backslash & & & & \backslash\varepsilon_k/ \\ & \nwarrow & & \nearrow & \\ & & /\backslash\varepsilon_k/\backslash & & \\ & & \uparrow & & \\ & & \varepsilon_k & & \end{array}$$

where $\alpha_k \to \gamma_k$ means that γ_k dominates α_k. There are no other dominations.

Therefore, the natural symmetric tensor norms define six different sequences $\{\alpha_k\}_k$ of tensor norms (which we call "natural sequences"), with their corresponding associated polynomial ideals. We denote by η_k the tensor norm $/\pi_k\backslash$, which is predual to the space of k-homogeneous extendible polynomials. Then we have $\eta_k' = \backslash\varepsilon_k/$, $\backslash\eta_k/ = \backslash/\pi_k\backslash/$ and $/\eta_k'\backslash = /\backslash\varepsilon_k/\backslash$.

To prove the coherence of the sequence of polynomial ideals associated to the natural symmetric tensor norms we need the following:

Lemma 3.1.34. *For each k, let α_k be a finitely generated k-fold symmetric tensor and suppose that $\alpha_{k-1}(\sum_j \gamma_j(a)\gamma_j^{k-1}) \leq c_k\alpha_k(\sum_j \gamma_j^k)\|a\|$ for every $\sum_j \gamma_j^k \in \bigotimes_{\alpha_k}^{k,s} E'$ and $a \in E$. Then the same inequality holds for the sequences $\{/\alpha_k\backslash\}_k$ and $\{\backslash\alpha_k/\}_k$.*

Proof. For $\{\backslash\alpha_k/\}_k$: if $\mathfrak{A}_k^{max}$ is the maximal ideal associated to α_k, the identity $\backslash\alpha_k/ = (/\alpha_k'\backslash)'$ and the representation theorem for maximal polynomial ideals [FH02, Section 3.2] show that the maximal polynomial ideal $\mathfrak{B}_k$ associated to $\backslash\alpha_k/$ at E is $\left(\bigotimes_{/\alpha_k'\backslash}^{k,s} E\right)'$. Thus it consists of all k-homogeneous polynomials on E which extend to α_k'-continuous polynomials on $\ell_\infty(B_{E'})$, that is,

$$\mathfrak{B}_k(E) = \{P \in \mathcal{P}^k(E) : P \text{ extends to a polynomial } \tilde{P} \in \mathfrak{A}_k^{max}(\ell_\infty(B_{E'}))\},$$

and the norm of P in $\mathfrak{B}_k$ is given by the infimum of the $\mathfrak{A}_k^{max}$-norms of these extensions. By Lemma 3.1.27, we may fix variables on the ideals $\mathfrak{A}_k^{max}$'s, and thus it is easy to see that we may fix variables for polynomials in the ideals $\mathfrak{B}_k$'s and with the same inequality of norms. Using again Lemma 3.1.27, we obtain the desired result for $\{\backslash\alpha_k/\}_k$.

For $\{/\alpha_k\backslash\}_k$: we denote

$$i_k = \otimes^k i : \big(\otimes^{k,s} E', /\alpha_k\backslash\big) \overset{1}{\hookrightarrow} \big(\otimes^{k,s} \ell_\infty(B_{E''}), \alpha_k\big),$$

and

$$\tilde{j} : \ell_1(B_{E''}) \overset{1}{\twoheadrightarrow} E.$$

Then, if $\mathfrak{a} \in \ell_1(B_{E''})$ such that $\tilde{j}(\mathfrak{a}) = a$ and $\|\mathfrak{a}\|_{\ell_1(B_{E''})} = \|a\|$,

$$\begin{aligned} /\alpha_{k-1}\backslash\Big(\sum_j \gamma_j(a)\gamma_j^{k-1}\Big) &= \alpha_{k-1}\Big(i_{k-1}\Big(\sum_j \gamma_j(a)\gamma_j^{k-1}\Big)\Big) = \alpha_{k-1}\Big(\sum_j i(\gamma_j)(\mathfrak{a})i(\gamma_j)^{k-1}\Big) \\ &\leq c_k\alpha_k\Big(\sum_j i(\gamma_j)^k\Big)\|\mathfrak{a}\|_{\ell_1(B_{E''})} = c_k/\alpha_k\backslash\Big(\sum_j \gamma_j^k\Big)\|a\|. \end{aligned}$$

$\square$

The following result is a particular case of Lemma 4.1.15:

Lemma 3.1.35. *For each k, let α_k be a finitely generated k-fold symmetric tensor and suppose that $\alpha_{k+1}(\sigma(s \otimes \gamma)) \leq c_k \alpha_k(s)\|\gamma\|$ for every $s \in \bigotimes_{\alpha_k}^{k,s} E'$ and $\gamma \in E'$. Then the same inequality holds for the sequences $\{/\alpha_k\backslash\}_k$ and $\{\backslash\alpha_k/\}_k$.*

As a consequence, since π_k and ε_k satisfy the conditions on previous lemmas, we can use them together with Lemmas 3.1.27 and 3.1.28 to show the following:

Theorem 3.1.36. *Let $\{\alpha_k\}_k$ be any of the natural sequences of symmetric tensor norms. Then the sequences $\{\mathfrak{A}_k^{max}\}_k$ and $\{\mathfrak{A}_k^{min}\}_k$ of maximal and minimal ideals associated to $\{\alpha_k\}_k$ are coherent.*

In the next chapter we will prove more properties of sequences of ideals associated to natural sequences of tensor norms.

3.2 Holomorphic mappings of bounded type associated to a coherent sequence

The space of holomorphic mappings of bounded type $H_b(E, F)$ is, in some sense, associated to the sequence $\{\mathcal{P}^k(E, F)\}_k$ of all homogeneous polynomials. Analogously, we can define the space of holomorphic functions of bounded type associated to any coherent polynomial sequence:

Definition 3.2.1. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a coherent sequence of polynomial ideals at (E, F). We define the space of $\mathfrak{A}$-entire functions of bounded type by*

$$H_{b\mathfrak{A}}(E, F) = \left\{ f \in H(E, F) \ : \frac{d^k f(0)}{k!} \in \mathfrak{A}_k(E, F) \ \textit{and} \ \lim_{k\to\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k}^{\frac{1}{k}} = 0 \right\}.$$

We define in $H_{b\mathfrak{A}}(E, F)$ the seminorms $\{p_R\}_{R>0}$,

$$p_R(f) = \sum_{k=0}^{\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k(E,F)} R^k,$$

for $f \in H_{b\mathfrak{A}}(E, F)$. Note that a function $f \in H(E, F)$ is in $H_{b\mathfrak{A}}(E, F)$ if and only if $\frac{d^k f(0)}{k!}$ belongs to $\mathfrak{A}_k(E, F)$ for every k and $p_R(f) < \infty$ for every R.

Proposition 3.2.2. *Suppose that $\mathfrak{A}$ is a coherent sequence of Banach ideals of homogeneous polynomials. Then $\Big(H_{b\mathfrak{A}}(E, F), \{p_n\}_{n\in\mathbb{N}}\Big)$ is a Fréchet space. Moreover, for each $f \in H_{b\mathfrak{A}}(E, F)$, the partial sums of the Taylor series expansion of f about the origin converges to f in $H_{b\mathfrak{A}}(E, F)$.*

Proof. We only prove that $H_{b\mathfrak{A}}(E, F)$ is complete. Let $(f_n)_n$ be a Cauchy sequence in $H_{b\mathfrak{A}}(E, F)$. Then $(f_n)_n$ is a Cauchy sequence in $H_b(E, F)$ and therefore there exists $f \in H_b(E, F)$ which is its limit. Also, for each k, $(\frac{d^k f_n(0)}{k!})_n$ is a Cauchy sequence in $\mathfrak{A}_k(E, F)$. Thus, $\frac{d^k f(0)}{k!}$ belongs to $\mathfrak{A}_k(E, F)$ because $\frac{d^k f(0)}{k!} = \lim_n \frac{d^k f_n(0)}{k!}$. Moreover, since $(p_R(f_n))_n$ is a Cauchy sequence, for each $R, \varepsilon > 0$, there is n_0 such that $p_R(f_n) \leq p_R(f_{n_0}) + \varepsilon$, for every $n \geq n_0$. Then for every $N \in \mathbb{N}$,

$$\sum_{k=0}^{N} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k(E,F)} R^k = \lim_n \sum_{k=0}^{N} \left\| \frac{d^k f_n(0)}{k!} \right\|_{\mathfrak{A}_k(E,F)} R^k \leq \lim_n p_R(f_n) \leq p_R(f_{n_0}) + \varepsilon$$

Therefore, $p_R(f) < \infty$ and thus f is in $H_{b\mathfrak{A}}(E, F)$. It is also clear that $f_n \to f$ in $H_{b\mathfrak{A}}(E)$.

Moreover, for each $f \in H_{b\mathfrak{A}}(E, F)$, and $R > 0$, $\sum_{k\geq N} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k(E,F)} R^k \to 0$ as $N \to \infty$, which means that the Taylor series of f about the origin converges to f in $H_{b\mathfrak{A}}(E, F)$. ∎

Although the definition of $H_{b\mathfrak{A}}(E,F)$ involves the derivatives at 0, the same condition holds for the derivatives at any point $a \in E$, as the following lemma shows.

Lemma 3.2.3. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a coherent sequence at (E,F) and $x \in E$. Then*

$$\begin{array}{rccc} \tau_x : & H_{b\mathfrak{A}}(E,F) & \to & H_{b\mathfrak{A}}(E,F) \\ & f & \mapsto & \tau_x f = f(x+\cdot) \end{array}$$

is a continuous operator. In particular, for all $x \in E$,

$$\lim_{k\to\infty} \Big\| \frac{d^k f(x)}{k!} \Big\|_{\mathfrak{A}_k(E,F)}^{\frac{1}{k}} = 0.$$

Proof. Take $f = \sum_{k=0}^{\infty} P_k \in H_{b\mathfrak{A}}(E,F)$ and $x \in E$. Then $P_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} (P_k)_{x^{k-j}}(y)$.

Thus $\tau_x f = \displaystyle\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (P_k)_{x^{k-j}}$. Using that the sequence is coherent it is easy to see that this series converges absolutely:

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \big\|(P_k)_{x^{k-j}}\big\|_{\mathfrak{A}_j(E,F)} \leq \sum_{k=0}^{\infty} (1 + C\|x\|)^k \|P_k\|_{\mathfrak{A}_k(E,F)} = p_{1+C\|x\|}(f). \tag{3.3}$$

So we can reverse the order of summation to obtain that $\frac{d^j \tau_x f(0)}{j!} = \sum_{k=j}^{\infty} \binom{k}{j} (P_k)_{x^{k-j}}$. Thus we have that

$$\begin{aligned} p_R(\tau_x f) &= \sum_{j=0}^{\infty} R^j \Big\| \frac{d^j \tau_x f(0)}{j!} \Big\|_{\mathfrak{A}_j(E,F)} \leq \sum_{j=0}^{\infty} R^j \sum_{k=j}^{\infty} \binom{k}{j} \big\|(P_k)_{x^{k-j}}\big\|_{\mathfrak{A}_j(E,F)} \\ &= \sum_{k=0}^{\infty} \|P_k\|_{\mathfrak{A}_k(E,F)} \sum_{j=0}^{k} \binom{k}{j} R^j (C\|x\|)^{k-j} \leq p_{R+C\|x\|}(f). \end{aligned}$$

Therefore $\tau_x f \in H_{b\mathfrak{A}}(E,F)$ and τ_x is continuous. $\square$

As we already mentioned, for each coherent sequence $\mathfrak{A}$ we can construct the space of $\mathfrak{A}$-entire functions of bounded type. Thus there are plenty of examples. We just present now the following examples of spaces of holomorphic functions of bounded type which were already defined in the literature and can be seen as particular cases of the above definition.

Example 3.2.4. Let $\mathfrak{A}$ be the following coherent sequence: $\mathfrak{A}_1 = \mathcal{L}$, $\mathfrak{A}_k = \mathcal{P}^k$, $k > 1$. Then $H_{b\mathfrak{A}}(E,F) = H_b(E,F)$ the space of entire functions of bounded type from E to F.

Example 3.2.5. If $\mathfrak{A}$ is the sequence of nuclear polynomial ideals then $H_{b\mathfrak{A}}(E,F)$ is the space of nuclearly entire functions of bounded type $H_{Nb}(E,F)$ defined by Gupta and Nachbin (see [Din99, Gup70]).

Example 3.2.6. If $\mathfrak{A}$ is the sequence of integral polynomial ideals then $H_{b\mathfrak{A}}(E)$ is the space of integral entire functions of bounded type $H_{bI}(E)$ defined by Dimant, Galindo, Maestre and Zalduendo in [DGMZ04].

Example 3.2.7. Suppose $\mathfrak{A}$ is the sequence of extendible polynomials, that is, $\mathfrak{A}_k(E) = \mathcal{P}_e^k(E)$, $k \geq 1$. Then an application of [Car01, Proposition 14] gives that $H_{b\mathfrak{A}}(E)$ is the space of all $f \in H(E)$ such that, for any Banach space $G \supset E$, there is an extension $\tilde{f} \in H_b(G)$ of f.

Example 3.2.8. Suppose $\mathfrak{A}$ is the sequence of approximable polynomials, $\mathfrak{A}_k(E) = \mathcal{P}^k_A(E)$, $k \geq 1$. Then $H_{b\mathfrak{A}}(E)$ is the space $H_{bc}(E)$ considered in [AB99, Din83].

Example 3.2.9. If $\mathfrak{A}$ is the sequence of weakly continuous on bounded sets polynomial ideals then $H_{b\mathfrak{A}}(E)$ is the space of weakly uniformly continuous holomorphic functions of bounded type $H_{bw}(E)$ defined by Aron in [Aro79].

Remark 3.2.10. Note that we have defined $\mathfrak{A}$-entire functions of bounded type as entire functions that have infinite "$\mathfrak{A}$-radius" of convergence at zero (and thus at every point). On the other hand, Hollstein [Hol86] defined, given an operator ideal $\mathfrak{C}$ the sequence of polynomial ideals $[\mathfrak{C}] = \{\mathcal{P}^k \circ \mathfrak{C}\}$. Then he defined the space of entire functions $H_b^{[\mathfrak{C}]}$, as the entire functions having positive $[\mathfrak{C}]$-radius of convergence at zero. The two definitions are different, indeed, Dineen showed an example of an entire function of bounded type, $f \in H_b(E)$ and $r > 0$ such that f has "nuclear radius" of convergence r (that f is holomorphic of nuclearly bounded type on rB_E, see Section 3.2.6), but found $x \in E$ such that $d^2 f(x) \notin \mathcal{P}^2_N(E)$ [Din71a, Example 11] (in particular $H_b^{[N]}(E) \neq H_{Nb}(E)$).

Dineen also found an example of an entire function of bounded type on a Hilbert space E, $f \in H_b(E)$, such that $\limsup_{n\to\infty} \|\frac{d^n f(x)}{n!}\|_N^{\frac{1}{n}} < \infty$ for every $x \in E$ (that is, f is locally of nuclear bounded type, or, at each point $x \in E$ there exist $r > 0$ such that f belongs to $H_{Nb}\big(B(x,r)\big)$, see Section 3.2.6) but f is not an entire function of nuclear type because $\lim_{n\to\infty} \|\frac{d^n f(0)}{n!}\|_N^{\frac{1}{n}} = 1$, [Din71a, Example 9].

3.2.1 Schauder decompositions

Recall that a sequence of Banach spaces $(E_n, \|\cdot\|_n)$ is a **Schauder decomposition** of a Fréchet space E if the following two conditions hold:

1. Each $x \in E$ can be written in a unique way as $x = \sum_n x_n$, with $x_n \in E_n$ for all n.
2. The projections $p_m : E \to E_n$, given by $p_m(\sum_{n=1}^\infty x_n) = \sum_{n=1}^m x_n$, are continuous.

Galindo, Maestre and Rueda [GMR00] introduced the concept of R-Schauder decompositions. Let us recall their definition.

Definition 3.2.11. *For $0 < R \leq \infty$, we say that a sequence of Banach spaces $(E_n, \|\cdot\|_n)$ is an* R**-Schauder decomposition** *of a Fréchet space E if it is a Schauder decomposition that satisfies the condition: for every sequence $(x_n)_n$, with $x_n \in E_n$, the series $\sum_{n=1}^\infty x_n$ converges in E if and only if $\limsup_n \|x_n\|_n^{1/n} \leq \frac{1}{R}$.*

The following is immediate from the definitions:

Proposition 3.2.12. *For every Banach spaces E and F, $\{\mathfrak{A}_k(E,F)\}_k$ is an ∞-Schauder decomposition of $H_{b\mathfrak{A}}(E,F)$.*

We can deduce some other consequences from the results on R-Schauder decompositions of the article [GMR00], combined with known facts about some ideals.

Proposition 3.2.13. *Let $\mathfrak{A}$ be a coherent sequence of polynomial ideals and let E and F be Banach spaces. Then:*

(a) $H_{b\mathfrak{A}}(E,F)$ is reflexive if and only if $\mathfrak{A}_k(E,F)$ is reflexive, for all k.

(b) *If E is Asplund, $H_{Nb}(E,F)$ is topologically isomorphic to $H_{bI}(E,F)$.*

(c) *$H_{b\mathfrak{A}}(E,F)$ contains copy of c_0 if and only if there exists $k \in \mathbb{N}$ such that $\mathfrak{A}_k(E,F)$ contains copy of c_0.*

Proof. The first item follow from [GMR00, Theorems 1 and 8].

The next item is a consequence of [GMR00, Theorem 1] and [CD00, Theorem 1.4].

The last item derive from [GMR00, Lemma 6] and [DD98, Theorem 4]. □

3.2.2 Weakly differentiable sequences and convolution

In this section we define the concept of weakly differentiable sequence which will be useful for the study of convolution operators on the space $H_{b\mathfrak{A}}(E)$ of scalar valued functions. The following convolution product was defined on the dual space to $H_b(E)$ by Aron, Cole and Gamelin in [ACG91]: if $\varphi, \psi \in H_b(E)'$, then $\varphi * \psi \in H_b(E)'$ is the linear functional given by

$$\varphi * \psi(f) = \psi(x \mapsto \varphi(\tau_x f)).$$

In this section we also define and investigate an analogous convolution product in $H_{b\mathfrak{A}}(E)'$.

We prove that this convolution product is well defined in a few steps. We know from Lemma 3.2.3 that if $\mathfrak{A}$ be a coherent sequence at E and $x \in E$. Then

$$\begin{array}{rccc} \tau_x : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E) \\ & f & \mapsto & \tau_x f = f(x + \cdot) \end{array}$$

is a continuous operator. Thus we are able to define:

Definition 3.2.14. *Let $\varphi \in H_{b\mathfrak{A}}(E)'$ and let $f \in H_{b\mathfrak{A}}(E)$. We define $\varphi * f : E \to \mathbb{C}$ as follows,*

$$\varphi * f(x) = \varphi \circ \tau_x(f) = \varphi(f(x + \cdot)).$$

Therefore, the desired convolution product can be rewritten as $\psi * \varphi(f) = \psi(\varphi * f)$. For this to be well defined, we must show that $\varphi * f \in H_{b\mathfrak{A}}(E)$ and that $f \to \varphi * f$ is continuous. We will need the additional condition of weakly differentiability of the sequence $\mathfrak{A}$ to achieve this. At the end of this section we will prove that there are plenty of weakly differentiable sequences of polynomial ideals. When a sequence of polynomial ideals is defined in both the scalar and vector valued case, one may consider the following property: "for every $P \in \mathfrak{A}_k(E)$, the mapping $x \mapsto P_{x^l}$ belongs to the space of vector-valued polynomials $\mathfrak{A}_l(E; \mathfrak{A}_{k-l}(E))$". This would mean that the differential of a polynomial in $\mathfrak{A}$ is also a polynomial in $\mathfrak{A}$. We consider a similar but less restrictive property, which makes sense when the polynomial ideals are only defined for the scalar case and that could be read as "the differential of a polynomial in $\mathfrak{A}$ is weakly in $\mathfrak{A}$". More precisely, we have:

Definition 3.2.15. *Let $\mathfrak{A}$ be a coherent sequence of polynomial ideals and let E be a Banach space. We say that $\mathfrak{A}$ is weakly differentiable (at E) if there exists a constant K such that, for $P \in \mathfrak{A}_k(E)$ and $\varphi \in \mathfrak{A}_{k-l}(E)'$, the mapping $x \mapsto \varphi(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and*

$$\Big\| x \mapsto \varphi(P_{x^l}) \Big\|_{\mathfrak{A}_l(E)} \leq K^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Since $\frac{d^{k-l}P}{(k-l)!}(x) = \binom{k}{l} P_{x^l}$, the previous condition is equivalent to say that $\varphi \circ \frac{d^{k-l}P}{(k-l)!}$ belongs to $\mathfrak{A}_l(E)$ and

$$\Big\|\varphi \circ \frac{d^{k-l}P}{(k-l)!}\Big\|_{\mathfrak{A}_l(E)} \leq \binom{k}{l} K^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

This is what we mean by saying that the differential is weakly in $\mathfrak{A}$ and what suggested our terminology.

We will also see in Section 4.1 that the weakly differentiability of a sequence may be seen as a property which is dual to being close under pointwise multiplication between homogeneous polynomials of the sequence.

Lemma 3.2.16. *For each $\varphi \in H_{b\mathfrak{A}}(E)'$ and $k \geq 1$, there exist constants $c, r > 0$ such that $\|\varphi|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \leq cr^k$ for every $P \in \mathfrak{A}_k(E)$.*

Proof. Indeed since φ is continuous, there are constants $c, r > 0$ such that $|\varphi(g)| \leq cp_r(g)$, for every $g \in H_{b\mathfrak{A}}(E)$. In particular, $|\varphi(P)| \leq cp_r(P) = cr^k \|P\|_{\mathfrak{A}_k(E)}$. Then $\|\varphi|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \leq cr^k$, for every $k \geq 1$. □

Theorem 3.2.17. *Let $\mathfrak{A}$ be a weakly differentiable coherent sequence at E. For each $\varphi \in H_{b\mathfrak{A}}(E)'$, the following operator is well defined and continuous:*

$$\begin{array}{rccc} T_\varphi : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E) \\ & f & \mapsto & \varphi * f \end{array}$$

Proof. Take $f = \sum_{k=0}^\infty P_k \in H_{b\mathfrak{A}}(E)$ and $x \in E$. Then

$$\varphi \circ \tau_x(f) = \sum_{k=0}^\infty \sum_{j=0}^k \binom{k}{j} \varphi\big((P_k)_{x^j}\big) = \sum_{j=0}^\infty \sum_{k=j}^\infty \binom{k}{j} \varphi\big((P_k)_{x^j}\big),$$

since, using Lemma 3.2.16 and the coherence of $\mathfrak{A}$ it is easy to see that this series is absolutely convergent.

Let $Q_l(x) = \sum_{k=l}^\infty \binom{k}{l} \varphi\big((P_k)_{x^l}\big)$. Then $\varphi \circ \tau_x(f) = \sum_{l=0}^\infty Q_l(x)$. We will show that Q_l belongs to $\mathfrak{A}_l(E)$ and that $\sum_{l=0}^\infty Q_l$ is in $H_{b\mathfrak{A}}(E)$. To prove this it suffices to show that the series $\sum_{k=l}^\infty \binom{k}{l} \Big\| x \mapsto \varphi\big((P_k)_{x^l}\big)\Big\|_{\mathfrak{A}_l(E)}$ converges and that for every $R > 0$, the series $\sum_{l=0}^\infty R^l \Big\| \sum_{k=l}^\infty \binom{k}{l} x \mapsto \varphi\big((P_k)_{x^l}\big)\Big\|_{\mathfrak{A}_l(E)}$ also converges:

$$\begin{aligned} \sum_{l=0}^\infty R^l \Big\| \sum_{k=l}^\infty \binom{k}{l} x \mapsto \varphi\big((P_k)_{x^l}\big)\Big\|_{\mathfrak{A}_l(E)} &\leq \sum_{l=0}^\infty R^l \sum_{k=l}^\infty \binom{k}{l} \Big\| x \mapsto \varphi\big((P_k)_{x^l}\big)\Big\|_{\mathfrak{A}_l(E)} \\ &\leq \sum_{l=0}^\infty R^l \sum_{k=l}^\infty \binom{k}{l} \|\varphi|_{\mathfrak{A}_{k-l}(E)}\|_{\mathfrak{A}_{k-l}(E)'} \|P_k\|_{\mathfrak{A}_k(E)} \\ &\leq c \sum_{k=0}^\infty \|P_k\|_{\mathfrak{A}_k(E)} K^k \sum_{l=0}^k \binom{k}{l} (R)^l r^{k-l} \\ &= cp_{K(R+r)}(f), \end{aligned}$$

where in the last inequality we used Lemma 3.2.16 and reversed the order of summation. Therefore $T_\varphi(f)$ belongs to $H_{b\mathfrak{A}}(E)$ and $p_R(T_\varphi(f)) \leq cp_{K(R+r)}(f)$, that is, $T_\varphi \in \mathcal{L}(H_{b\mathfrak{A}}(E), H_{b\mathfrak{A}}(E))$. □

An operator $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ that commutes with translations is said to be a convolution operator, that is T is a convolution operator if $T \circ \tau_x = \tau_x \circ T$ for all $x \in E$. We have the following characterization of convolution operators:

Corollary 3.2.18. *Let $\mathfrak{A}$ be a coherent sequence and $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ a convolution operator. Then there exist $\varphi \in H_{b\mathfrak{A}}(E)'$ such that $Tf = \varphi * f$ for every $f \in H_{b\mathfrak{A}}(E)$. The converse is true for weakly differentiable sequences.*

Moreover, if finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k, then each convolution operator T determines a unique function $\varphi \in H_{b\mathfrak{A}}(E)'$.

Proof. Let $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ be a convolution operator and let $\varphi = T \circ \delta_0$. Then $Tf(x) = \tau_x(Tf)(0) = T(\tau_x f)(0) = \varphi \circ \tau_x(f) = \varphi * f(x)$.

The converse for weakly differentiable sequences is a consequence of the above theorem since, for each $\varphi \in H_{b\mathfrak{A}}(E)'$, the mapping $f \mapsto \varphi * f$ is a convolution operator.

Moreover, if $T(f) = \psi * f = \varphi * f$ for every $f \in H_{b\mathfrak{A}}(E)$ then $\psi(\gamma^n) = \psi * \gamma^n(0) = T(\gamma^n)(0) = \varphi(\gamma^n)$ for every $\gamma \in E'$ and $n \in \mathbb{N}$ and thus density of finite type polynomials imply that $\varphi = \psi$. $\square$

Corollary 3.2.19. *Suppose that $\mathfrak{A}$ be a weakly differentiable coherent sequence and finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k then $\varphi \mapsto T_\varphi$ is a vector space isomorphism between $H_{b\mathfrak{A}}(E)'$ and the convolution operators on $H_{b\mathfrak{A}}(E)$.*

As a consequence of Theorem 3.2.17 we also have:

Corollary 3.2.20. *Suppose that $\mathfrak{A}$ is a weakly differentiable coherent sequence. If $\varphi \in H_{b\mathfrak{A}}(E)'$, then the application*

$$\begin{array}{rccc} M_\varphi : & H_{b\mathfrak{A}}(E)' & \to & H_{b\mathfrak{A}}(E)' \\ & \psi & \mapsto & \psi * \varphi, \end{array}$$

where $\psi * \varphi(f) = \psi(\varphi * f)$, is a continuous linear operator when we consider the strong dual topology on $H_{b\mathfrak{A}}(E)'$.

Proof. Just note that M_φ is the transpose of T_φ. $\square$

This allows us to define the desired product:

Definition 3.2.21. *Let $\mathfrak{A}$ is a weakly differentiable coherent sequence. For $\varphi, \psi \in H_{b\mathfrak{A}}(E)'$, the* ***convolution product*** *$\varphi * \psi \in H_{b\mathfrak{A}}(E)'$ is defined by*

$$\varphi * \psi(f) = \psi(x \mapsto \varphi \circ \tau_x(f)) = \psi(\varphi * f),$$

for $f \in H_{b\mathfrak{A}}(E)$.

Now we show that, many sequences of polynomial ideals are weakly differentiable.

Example 3.2.22. The following sequences are weakly differentiable:

(a) $\mathfrak{A} = \{\mathcal{P}^k\}_k$: if $P \in \mathcal{P}^k(E)$ and $\varphi \in \mathcal{P}^{k-l}(E)'$ then it is clear that $x \mapsto \varphi\big(P_{x^l}\big) \in \mathcal{P}^l(E)$ and $\big\|x \mapsto \varphi\big(P_{x^l}\big)\big\|_{\mathcal{P}^l(E)} \le e^l \|\varphi\|_{\mathcal{P}^{k-l}(E)'} \|P\|_{\mathcal{P}^k(E)}$.

(b) $\mathfrak{A} = \{\mathcal{P}_I^k\}_k$: this will be proved in Corollary 4.1.19.

(c) $\mathfrak{A} = \{\mathcal{P}_e^k\}_k$: if $P \in \mathcal{P}_e^k(E)$ and $\varphi \in \mathcal{P}_e^{k-l}(E)'$ then $Q(x) = \varphi\big(P_{x^l}\big)$ is in $\mathcal{P}^l(E)$. Let $E \overset{J}{\hookrightarrow} G$ and $\tilde{P}$ an extension of P to G. Then $\tilde{Q}(y) = \varphi\big(\tilde{P}_{y^l} \circ J\big)$ is an extension of Q to G, and thus Q is extendible. Moreover, since $|\tilde{Q}(y)| \leq e^l \|y\|^l \|\varphi\| \|\tilde{P}\|$, it follows that $\|Q\|_e \leq e^l \|\varphi\| \|P\|_e$.

(d) $\mathfrak{A} = \{\mathcal{P}_w^k\}_k$: it is known (see [Din99, Proposition 2.6]) that if $P \in \mathcal{P}_w^k(E)$ then $d^{k-l}P$ is weakly continuous. Thus $x \mapsto \varphi\big(P_{x^l}\big) \in \mathcal{P}_w^l(E)$ and and has norm $\leq e^l \|\varphi\|_{\mathcal{P}^{k-l}(E)'} \|P\|_{\mathcal{P}^k(E)}$.

We finish this section with sets of examples of weakly differentiable sequences. The first one deals with ideals associated to tensor norms and the second one with composition ideals. See also Lemma 4.2.1 for some examples on minimal ideals.

Lemma 3.2.23. *Let $\mathfrak{A}$ be the sequence of maximal polynomial ideals associated to a sequence of symmetric tensor norms $\{\alpha_k\}_k$. If $\mathfrak{A}$ is weakly differentiable, then the same is true for the sequences of maximal polynomial ideals associated to $\{\backslash\alpha_k/\}_k$ and to $\{/\alpha_k\backslash\}_k$.*

Proof. Let us denote $\beta_k = \alpha_k'$. For $\mathfrak{A}$ the sequence of maximal polynomial ideals associated to $\{\backslash\alpha_k/\}_k$ we have that P belongs to $\mathfrak{A}_k(E)$ if and only if $P \in \Big(\otimes_{/\beta_k\backslash}^{k,s} E\Big)'$, and we can proceed just as we did with the ideal of extendible polynomials (note that polynomials in $\mathfrak{A}_k(E)$ are those that extends to a β_k-continuous polynomial on $\ell_\infty(B_{E'})$).

For the sequence of maximal ideals associated to $\{/\alpha_k\backslash\}_k$, P belongs to $\mathfrak{A}_k(E)$ if and only if $\widetilde{P} = P \circ q_k$ belongs to $\Big(\otimes_{\beta_k}^{k,s} \ell_1(B_E)\Big)' = \mathfrak{A}_k(\ell_1(B_E))$, where q_k is the metric projection $\otimes_{\beta_k}^{k,s} \ell_1(B_E) \twoheadrightarrow \otimes_{\backslash\beta_k/}^{k,s} E$. Also, transposing q_k we obtain a metric injection $\Big(\otimes_{\backslash\beta_k/}^{k,s} E\Big)' \hookrightarrow \Big(\otimes_{\beta_k}^{k,s} \ell_1(B_E)\Big)'$. If we take $\varphi \in \Big(\otimes_{\backslash\beta_k/}^{k,s} E\Big)'$, we can choose a Hahn-Banach extension ψ of φ on $\otimes_{\beta_k}^{k,s} \ell_1(B_E)$.

Now, if $Q(x) = \varphi\big(P_{x^l}\big)$, we have

$$Q \circ q(z) = \varphi(P_{q(z)^l}) = \psi\left((P \circ q)_{z^l}\right),$$

which belongs to $\Big(\otimes_{\beta_k}^{k-l,s} \ell_1(B_E)\Big)'$ because $\mathfrak{A}$ is weakly differentiable. But this means that Q belongs to $\mathfrak{A}_{k-l}(E)$. ☐

In the previous proof, we only used that $\mathfrak{A}$ is weakly differentiable in spaces of the form $\ell_\infty(I)$ (for $\{\backslash\alpha_k/\}_k$) and $\ell_1(J)$ ($\{/\alpha_k\backslash\}_k$), where I and J are some index sets.

From the previous lemma and examples (a) and (b) above we have,

Corollary 3.2.24. *The sequence of maximal polynomial ideals associated to any of the natural sequences is weakly differentiable.*

Proposition 3.2.25. *Let $\{\mathfrak{A}_k\}_k$ be a weakly differentiable sequence and $\mathfrak{C}$ is a normed operator ideal. Then $\{\mathfrak{A}_k \circ \mathfrak{C}\}_k$ is weakly differentiable.*

Proof. Let $P = RT \in \mathfrak{A}_k \circ \mathfrak{C}(E)$, with $T \in \mathfrak{C}(E, E_1)$ and $R \in \mathfrak{A}_k(E_1)$. Take $\varphi \in \mathfrak{A}_{k-l} \circ \mathfrak{C}(E)'$ and define $\psi \in \mathfrak{A}_{k-l}(E_1)'$ by $\psi(Q) = \varphi(QT)$. Note that $\|\psi\|_{\mathfrak{A}_{k-l}(E_1)'} \leq \|\varphi\|_{\mathfrak{A}_{k-l}\circ\mathfrak{C}(E)'} \|T\|_{\mathfrak{C}(E,E_1)}^{k-l}$. Clearly,

$$\varphi(P_{x^l}) = \varphi(R_{(Tx)^l} T) = \psi(R_{(Tx)^l}),$$

thus $x \mapsto \varphi(P_{x^l}) = \big[x \mapsto \psi(R_{x^l})\big] \circ T$ belongs to $\mathfrak{A}_l \circ \mathfrak{C}(E)$, because $\{\mathfrak{A}_k\}$ is weakly differentiable and $T \in \mathfrak{C}$. Moreover

$$\begin{aligned}\|x \mapsto \varphi(P_{x^l})\|_{\mathfrak{A}_l \circ \mathfrak{C}(E)} &= \|\big[x \mapsto \psi(R_{x^l})\big] \circ T\|_{\mathfrak{A}_l \circ \mathfrak{C}(E)} \leq \|x \mapsto \psi(R_{x^l})\|_{\mathfrak{A}_l(E_1)} \|T\|^l_{\mathfrak{C}(E,E_1)} \\ &\leq K^k \|\psi\|_{\mathfrak{A}_{k-l}(E_1)'} \|R\|_{\mathfrak{A}_k(E_1)} \|T\|^l_{\mathfrak{C}(E,E_1)} \\ &\leq K^k \|\varphi\|_{\mathfrak{A}_{k-l} \circ \mathfrak{C}(E)'} \|R\|_{\mathfrak{A}_k(E_1)} \|T\|^k_{\mathfrak{C}(E,E_1)}.\end{aligned}$$

Since this is true for every factorization of $P = RT$, we conclude that

$$\|x \mapsto \varphi(P_{x^l})\|_{\mathfrak{A}_l \circ \mathfrak{C}(E)} \leq K^k \|\varphi\|_{\mathfrak{A}_{k-l} \circ \mathfrak{C}(E)'} \|P\|_{\mathfrak{A}_k \circ \mathfrak{C}(E)}.$$

□

Example 3.2.26. *The ∞-factorable (∞-compact) polynomials form a sequence of normed ideals which is coherent and weakly differentiable since it is the composition of all the polynomials with the ∞-factorable (∞-compact) operators [Hol86]. We already knew that the sequence of ∞-factorable polynomials is weakly differentiable since it coincide with the extendible polynomials ([KR98, Car99]).*

Corollary 3.2.27. *If $\{\mathfrak{A}_k\}_k$ is a weakly differentiable sequence then $\{\mathfrak{A}_k^{min}\}_k$ is weakly differentiable.*

3.2.3 Coherent sequences of minimal ideals and duality

This subsection deals with duality questions for $H_{b\mathfrak{A}}$, when the ideals in the sequence are minimal. We will restrict ourselves to scalar valued functions. Based on duality properties for each space $\mathfrak{A}_k(E)$, we characterize the dual of $H_{b\mathfrak{A}}(E)$ as the space of holomorphic functions of exponential type $Exp_{\mathfrak{B}}(E')$ associated to some sequence $\mathfrak{B}$ of Banach spaces of polynomials.

Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a coherent sequence at E. The Borel transform $\beta : H_{b\mathfrak{A}}(E)' \to H(E')$ assigns to each element $\varphi \in H_{b\mathfrak{A}}(E)'$ the holomorphic function $\beta(\varphi) \in H(E')$, given by $\beta(\varphi)(\gamma) = \varphi(\exp \circ \gamma) = \varphi(e^\gamma)$.

If $\varphi \in \mathfrak{A}_k(E)'$, we have two natural ways to identify φ with an element in $H_{b\mathfrak{A}}(E)'$:

$$\begin{array}{ccc} H_{b\mathfrak{A}}(E) & \xrightarrow{\widetilde{\varphi}} & \mathbb{C} \\ f & \longmapsto & \varphi\left(\frac{d^k f(0)}{k!}\right) \end{array} \qquad \text{or} \qquad \begin{array}{ccc} H_{b\mathfrak{A}}(E) & \xrightarrow{\overline{\varphi}} & \mathbb{C} \\ f & \longmapsto & \varphi\left(d^k f(0)\right). \end{array}$$

Thus, the Borel transform induces two different "polynomial" Borel transforms: $\beta_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ where $\beta_k(\varphi) = \beta(\widetilde{\varphi})$ and $B_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ given by $B_k(\varphi) = \beta(\overline{\varphi})$. Note that for $\gamma \in E'$, $\beta_k(\varphi)(\gamma) = \varphi\left(\frac{\gamma^k}{k!}\right)$ and $B_k(\varphi)(\gamma) = \varphi\left(\gamma^k\right)$.

In the polynomial setting it is more common to use the mapping B_k than the mapping β_k. Moreover, it is not necessary to deal with holomorphic functions in order to define the polynomial Borel transform B_k. Indeed, for a Banach space of k-homogeneous polynomials $\mathfrak{A}_k(E)$, we can define $B_k : \mathfrak{A}_k(E)' \to \mathcal{P}^k(E')$ by $B_k(\varphi)(\gamma) = \varphi\left(\gamma^k\right)$, for every $\gamma \in E'$. Also, we can express the holomorphic Borel transform β in terms of the B_k's:

$$\beta(\varphi) = \sum_{k=0}^{\infty} \frac{B_k\big(\varphi|_{\mathfrak{A}_k(E)}\big)}{k!}.$$

Remark 3.2.28. *In the sequel, the expression*

$$\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$$

will always mean that the polynomial Borel transform $B_k : \mathfrak{A}_k(E)' \to \mathfrak{B}_k(E')$ is an isometric isomorphism.

The following lemma states that in order to have this duality, $\mathfrak{A}_k(E)$ must be "small":

Lemma 3.2.29. *If $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$, then finite type polynomials are dense in $\mathfrak{A}_k(E)$:*

$$\overline{\mathcal{P}_f^k(E)}^{\mathfrak{A}_k} = \mathfrak{A}_k(E).$$

Proof. Suppose there exists $P \in \mathfrak{A}_k(E) - \overline{\mathcal{P}_f^k(E)}^{\mathfrak{A}_k}$. Then there is $\varphi \in \mathfrak{A}_k(E)'$ such that $\varphi(P) = 1$ and $\varphi|_{\mathcal{P}_f^k(E)} \equiv 0$. For every $\gamma \in E'$, $\varphi(\gamma^k) = 0$ and then $B_k(\varphi)(\gamma) = 0$. Thus $B_k(\varphi) = 0$ and therefore $\varphi = 0$ in $\mathfrak{A}_k(E)$, which is a contradiction. □

Since the Taylor expansion about the origin of a function $f \in H_{b\mathfrak{A}}(E)$ converges in $H_{b\mathfrak{A}}(E)$, we have

Corollary 3.2.30. *Let $\{\mathfrak{A}_k(E)\}_k$ and $\{\mathfrak{B}_k(E')\}_k$ be coherent sequences such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$. Then finite type polynomials are dense in $H_{b\mathfrak{A}}(E)$.*

Example 3.2.31. We exhibit two simple situations of coherent sequences where the duality $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ holds.

First, if $\mathcal{P}_A^n(E)$ is the space of approximable n-homogeneous polynomials, then $\mathcal{P}_A^n(E)'$ is (isometrically) the space of integral polynomials $\mathcal{P}_I^n(E')$ [Din71a].

Second, if E' has the approximation property, the dual of $\mathcal{P}_N^n(E)$ coincides with $\mathcal{P}^n(E')$ [Gup70].

Note that in both cases, the sequence of dual spaces (i.e. $\{\mathcal{P}^n(E')\}_n$ and $\{\mathcal{P}_I(E')\}_n$ respectively) is coherent.

Remark 3.2.32. Now we use results from [Flo01] on minimal, maximal and dual (or adjoint) polynomial ideals to show how to obtain other examples in which the duality relation $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ holds.

Suppose $\mathfrak{A}_k$ is a minimal ideal and let α_k be its associated k-symmetric tensor norm. If E' has the bounded approximation property, then $\mathfrak{A}_k(E)$ identifies isometrically with $\otimes_{s,\alpha_k}^k E'$ and then $\mathfrak{A}_k(E)'$ is isometrically isomorphic to $\mathfrak{B}_k(E') = \mathfrak{A}_k^*(E')$ via the Borel transform B_k, see [Flo01, Corollary 4.3].

On the other hand, if we start with a maximal ideal $\mathfrak{B}_k$, let $\mathfrak{A}_k = (\mathfrak{B}_k^*)^{min}$. Again, if E' has the bounded approximation property, the Borel transform B_k is an isometric isomorphism between $\mathfrak{A}_k(E)'$ and $\mathfrak{B}_k(E')$.

The following proposition states that if the duals of $\mathfrak{A}_k(E)$ form a coherent sequence of spaces of polynomials, then $\{\mathfrak{A}_k(E)\}_k$ inherits the coherence.

Proposition 3.2.33. *Let $\{\mathfrak{B}_k\}_k$ be a coherent sequence at E' with constants C and D, and suppose $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. Then, $\{\mathfrak{A}_k(E)\}_k$ is a coherent sequence with constants D and C.*

Proof. First, observe that if $\xi \in E'$ and $a \in E$ then $(\xi^{k+1})_a = \xi(a)\xi^k$. Thus, for every $\psi \in \mathfrak{B}_k(E')$,

$$B_k^{-1}(\psi)((\xi^{k+1})_a) = \xi(a)B_k^{-1}(\psi)(\xi^k) = \xi(a)\psi(\xi) = (a\psi)(\xi) = B_{k+1}^{-1}(a\psi)(\xi^{k+1}).$$

This implies that, for every $P \in \mathcal{P}_f^{k+1}(E)$, $B_k^{-1}(\psi)(P_a) = B_{k+1}^{-1}(a\psi)(P)$ and

$$\|P_a\|_{\mathfrak{A}_k(E)} = \sup_{\|\psi\|_{\mathfrak{B}_k(E')}=1} |B_{k+1}^{-1}(a\psi)(P)| \le D\|a\|\|P\|_{\mathfrak{A}_{k+1}(E)}.$$

By the density result in Lemma 3.2.29, we obtain that for every $P \in \mathfrak{A}_{k+1}(E)$ and every $a \in E$, P_a belongs to $\mathfrak{A}_k(E)$ and $\|P_a\|_{\mathfrak{A}_k(E)} \le D\|a\|\|P\|_{\mathfrak{A}_{k+1}(E)}$.

To prove the second condition of coherence, note that if γ and ξ are in E' and $\psi \in \mathfrak{B}_{k+1}(E')$ we have, by the polarization formula,

$$\begin{aligned}
&B_{k+1}^{-1}(\psi)(\gamma\xi^k) = \\
&\quad = \frac{1}{2^{k+1}(k+1)!} \sum_{\varepsilon_1,\dots,\varepsilon_{k+1}=\pm1} \varepsilon_1 \cdots \varepsilon_{k+1} B_{k+1}^{-1}(\psi)\big((\varepsilon_1\gamma + (\varepsilon_2 + \cdots + \varepsilon_{k+1})\xi)^{k+1}\big) \\
&\quad = \frac{1}{2^{k+1}(k+1)!} \sum_{\varepsilon_1,\dots,\varepsilon_{k+1}=\pm1} \varepsilon_1 \cdots \varepsilon_{k+1} \psi\big(\varepsilon_1\gamma + (\varepsilon_2 + \cdots + \varepsilon_{k+1})\xi\big) \\
&\quad = \psi_\gamma(\xi) = B_k^{-1}(\psi_\gamma)(\xi^k).
\end{aligned}$$

This implies that if P is a finite type k-homogeneous polynomial on E, then $B_{k+1}^{-1}(\psi)(\gamma P) = B_k^{-1}(\psi_\gamma)(P)$. And thus, for every $P \in \mathcal{P}_f^k(E)$,

$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} = \sup_{\|\psi\|_{\mathfrak{B}_k(E')}=1} |B_k^{-1}(\psi_\gamma)(P)| \le C\|\gamma\|\|P\|_{\mathfrak{A}_k(E)}.$$

Therefore, again by Lemma 3.2.29, for every $P \in \mathfrak{A}_k(E)$ the polynomial γP belongs to $\mathfrak{A}_{k+1}(E)$ and $\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} \le C\|\gamma\|\|P\|_{\mathfrak{A}_k(E)}$. ◻

In order to study the dual of $H_{b\mathfrak{A}}(E)$, we need the following

Definition 3.2.34. *Let $\mathfrak{B} = \{\mathfrak{B}_k\}_k$ be a coherent sequence at E. We define the holomorphic functions of $\mathfrak{B}$-exponential type on E,*

$$Exp_{\mathfrak{B}}(E) = \Big\{f \in H(E) \ : \ d^kf(0) \in \mathfrak{B}_k(E) \text{ for all } k \text{ and } \limsup_{k\to\infty} \big\|d^kf(0)\big\|_{\mathfrak{B}_k}^{\frac{1}{k}} < \infty\Big\}.$$

A classical result of Gupta states that, for E' with the approximation property, the Borel transform defines a duality between the space of nuclearly entire functions of bounded type over E, $H_{Nb}(E)$, and the space of holomorphic mappings of exponential type on E', $Exp(E')$ [Gup70, Din99]. In an analogous way, we prove the following:

Proposition 3.2.35. *Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence and let $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. Then the Borel transform is a vector space isomorphism between $H_{b\mathfrak{A}}(E)'$ and $Exp_{\mathfrak{B}}(E')$.*

Proof. Let $\varphi \in H_{b\mathfrak{A}}(E)'$. Since φ is continuous, there are constants $c, R > 0$ such that $|\varphi(g)| \le cp_R(g)$, for every $g \in H_{b\mathfrak{A}}(E)$. In particular, for each k, if g belongs to $\mathfrak{A}_k(E)$, we get $|\varphi(g)| \le cR^k\|g\|_{\mathfrak{A}_k(E)}$. Then $\|\varphi_{|_{\mathfrak{A}_k(E)}}\|_{\mathfrak{A}_k(E)'} \le cR^k$, for every $k \ge 1$. Moreover, since $\frac{d^k\beta(\varphi)(0)}{k!}(\gamma) = \varphi_{|_{\mathfrak{A}_k(E)}}(\frac{\gamma^k}{k!})$ we have that $d^k\beta(\varphi)(0) = B_k\big(\varphi_{|_{\mathfrak{A}_k(E)}}\big)$. Then $\|d^k\beta(\varphi)(0)\|_{\mathfrak{B}_k(E')}^{\frac{1}{k}} = \|\varphi_{|_{\mathfrak{A}_k(E)}}\|_{\mathfrak{A}_k(E)'}^{\frac{1}{k}} \le c^{\frac{1}{k}}R$. Therefore, $\beta(\varphi) \in Exp_{\mathfrak{B}}(E')$.

The Borel transform β is injective as a consequence of Corollary 3.2.30. To see that it is also surjective, let $\psi \in Exp_{\mathfrak{B}}(E')$ and $A = \sup_k \|d^k\psi(0)\|_{\mathfrak{B}_k(E')}^{\frac{1}{k}}$. For each $g \in H_{b\mathfrak{A}}(E)$, we define

$$\varphi(g) = \sum_{k=0}^{\infty} B_k^{-1}\big(d^k\psi(0)\big)\Big(\frac{d^k g(0)}{k!}\Big).$$

Since

$$|\varphi(g)| \leq \sum_{k=0}^{\infty} \|d^k\psi(0)\|_{\mathfrak{B}_k(E')}\Big\|\frac{d^k g(0)}{k!}\Big\|_{\mathfrak{A}_k(E)} \leq \sum_{k=0}^{\infty} A^k\Big\|\frac{d^k g(0)}{k!}\Big\|_{\mathfrak{A}_k(E)} = p_A(g),$$

we have $\varphi \in H_{b\mathfrak{A}}(E)'$. Finally, simple computations show that $\beta(\varphi) = \psi$. □

3.2.4 Convolution operators and hypercyclicity

Let X be a Fréchet space. An operator $T : X \to X$ is **hypercyclic** if there exists $x \in X$ such that its orbit $\{T^n x : n \geq 1\}$ is dense in X.

The first example of an hypercyclic operator was given in 1929 by Birkhoff, who proved that translation operators in $\mathcal{H}(\mathbb{C})$ are hypercyclic:

Theorem 3.2.36 (Birkhoff, [Bir29])**.** *Let $0 \neq a \in \mathbb{C}$. Then there exists a function $g \in \mathcal{H}(\mathbb{C})$ such that the set $\{g(na + \cdot) : n \in \mathbb{N}\}$ is dense in $\mathcal{H}(\mathbb{C})$ (with the topology of uniform convergence on compact sets), that is, the operator $\tau_a : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ defined by $\tau_a(f) = f(a + \cdot)$ is hypercyclic.*

A related result was shown by MacLane in 1952:

Theorem 3.2.37 (MacLane, [Mac52])**.** *The differentiation operator on $\mathcal{H}(\mathbb{C})$ is hypercyclic.*

In a seminal work, Godefroy and Shapiro [GS91] showed that every convolution operator in $\mathcal{H}(\mathbb{C}^n)$ which is not a scalar multiple of identity is hypercyclic. In this way, they generalized the classical results of Birkhoff [Bir29] and MacLane [Mac52] mentioned above. Analogues of Godefroy and Shapiro's result for some particular spaces of holomorphic functions on Banach spaces are proved in [AB99, Pet01, Pet06].

Given a coherent sequence $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ of Banach spaces of k-homogeneous polynomials, we defined a Fréchet space $H_{b\mathfrak{A}}(E)$ of holomorphic function of bounded type associated to $\mathfrak{A}(E)$. In this subsection, under the duality conditions for the $\mathfrak{A}_k(E)$'s studied in the previous subsection, we prove a Godefroy-Sahpiro theorem for $H_{b\mathfrak{A}}(E)$. We obtain some results of [AB99, Pet01, Pet06] as particular cases. We prove that spaces of holomorphic functions generated by polynomial minimal ideals are covered by our settings, if the dual space E' has the approximation property. We will also consider polynomials of the Schatten-von Neumann class in the sense of [CKP92] and the associated space of holomorphic functions in the next subsection. We will use the characterization of convolution operators on the space of holomorphic functions $H_{b\mathfrak{A}}(E)$ obtained in subsection 3.2.2, the description of $H_{b\mathfrak{A}}(E)'$ given in Proposition 3.2.35 and the hypercyclicity criterion. This criterion is most commonly used tool to prove the hypercyclicity of linear operators. It was first proved by Kitai in her unpublished thesis, and some years later rediscovered by Gethner and Shapiro:

Theorem 3.2.38 (Hypercyclicity Criterion, [GS87, Kit82])**.** *Let X be a separable Fréchet space and T a continuous linear operator on X. Suppose that there are dense subsets X_0 and Y_0 of X, an increasing sequence (n_k) of positive integers and (possibly non-linear and discontinuous) mappings $S_{n_k} : Y_0 \to X$ such that*

(i) for every $x \in X_0$; $T^{n_k}x \to 0$,
(ii) for every $y \in Y_0$; $S_{n_k}y \to 0$,
(iii) for every $y \in Y_0$; $T_{n_k} \circ S_{n_k}y \to y$.
Then the operator T is hypercyclic.

A longstanding open problem in the theory of the dynamics of linear operators was if every hypercyclic operator satisfies the Hypercyclicity Criterion. Recently, De La Rosa and Read [dlRR09], constructed a Banach space X and a hypercyclic operator T which does not satisfy the Hypercyclicity Criterion.

We now prove the announced result about hypercyclicity of convolution operators. We follow the steps of the proof of [Pet01, Theorem 3.1].

Theorem 3.2.39 (The Godefroy-Shapiro Theorem for $H_{b\mathfrak{A}}$)**.** *Suppose that E' is separable. Let $\{\mathfrak{B}_k(E')\}_k$ be a coherent sequence and $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then, every convolution operator $T : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{A}}(E)$ which is not a scalar multiple of the identity is hypercyclic.*

Proof. By Corollary 3.2.18, there is a linear functional $\varphi \in H_{b\mathfrak{A}}(E)'$ which satisfies $T(f) = \varphi * f$ for every f. Since T is not a scalar multiple of the identity, it follows that φ is not a scalar multiple of δ_0.

Since E' is separable, by Corollary 3.2.30, $H_{b\mathfrak{A}}(E)$ is separable. Therefore, we can use the Hypercyclicity Criterion 3.2.38.

First, note that $\operatorname{span}\{e^\gamma : \gamma \in U\}$ is dense in $H_{b\mathfrak{A}}(E)$ for any nonempty open set $U \subset E'$. Indeed, if $\psi \in H_{b\mathfrak{A}}(E)'$ and $\psi(e^\gamma) = 0$ for every $\gamma \in U$, then $\beta(\psi) \equiv 0$ in U and we have $\beta(\psi) = 0$. This means that ψ is 0.

Also, the fact that φ is not a scalar multiple of δ_0 implies that $\beta(\varphi)$ is not a constant function. Indeed, if $\beta(\varphi)$ was constant then $\lambda = \varphi(1) = \beta(\varphi)(0) = \beta(\varphi)(\gamma) = \varphi(e^\gamma)$ for all $\gamma \in E'$. But, on the other hand, $\lambda = \lambda\delta_0(e^\gamma)$ for all $\gamma \in E'$ and we would have that $\varphi = \lambda\delta_0$.

We will now prove that T satisfies the Hypercyclicity Criterion. Let

$$V = \{\gamma \in E' : |\beta(\varphi)(\gamma)| < 1\} \qquad \text{and} \qquad W = \{\gamma \in E' : |\beta(\varphi)(\gamma)| > 1\}.$$

Then $V, W \subset E'$ are open sets, and they are nonempty. Indeed, if $W = \varnothing$ $(V = \varnothing)$ then $\beta(\varphi)$ $(\frac{1}{\beta(\varphi)})$ would be a nonconstant bounded entire function. Let

$$H_V(E) = \operatorname{span}\{e^\gamma,\ \gamma \in V\} \qquad \text{and} \qquad H_W(E) = \operatorname{span}\{e^\gamma,\ \gamma \in W\}.$$

As we have observed, $H_V(E)$ and $H_W(E)$ are dense in $H_{b\mathfrak{A}}(E)$.

For $\gamma \in V$,

$$T(e^\gamma) = \varphi * e^\gamma = \big[x \mapsto \varphi(\tau_x e^\gamma)\big] = \varphi(e^\gamma)e^\gamma = \beta(\varphi)(\gamma)e^\gamma.$$

Then $T(H_V(E)) \subset H_V(E)$. Also, $T^n(e^\gamma) = \beta(\varphi)(\gamma)^n e^\gamma$, and since $|\beta(\varphi)(\gamma)| < 1$ for $\gamma \in V$, we obtain that $T^n(f) \underset{n\to\infty}{\longrightarrow} 0$, for every $f \in H_V(E)$.

For $\gamma \in W$, let $S(e^\gamma) = \dfrac{e^\gamma}{\beta(\varphi)(\gamma)}$. Since $\{e^\gamma,\ \gamma \in W\}$ is linearly independent (see the proof of [AB99, Lemma 2.3]), we can linearly extend S to $H_W(E)$. Then $S(H_W(E)) \subset H_W(E)$ and $S^n(e^\gamma) = \dfrac{e^\gamma}{\beta(\varphi)(\gamma)^n}$. Thus $S^n(f) \underset{n\to\infty}{\longrightarrow} 0$, for every $f \in H_W(E)$, since $|\beta(\varphi)(\gamma)| > 1$ for $\gamma \in W$.

Finally, $TS(e^\gamma) = T\Big(\dfrac{e^\gamma}{\beta(\varphi)(\gamma)}\Big) = e^\gamma$ and therefore $TSf = f$ for all $f \in H_W(E)$.
By the Hypercyclicity Criterion, T is hypercyclic. $\square$

Now we apply the previous results to different spaces of holomorphic functions.

Example 3.2.40. In [AB99] the authors study differentiation operators in $H_{bc}(E)$, the space of holomorphic functions of compact bounded type on E (that is: $f = \sum P_n \in H_{bc}(E)$ whenever each P_n is an approximable n-homogeneous polynomial and $\|P_n\|^{\frac{1}{n}} \to 0$, where $\|\cdot\|$ denotes the usual norm). They show that if the differentiation operator is constructed from an entire function of exponential type on $\mathbb{C}$, then it is hypercyclic. This result is a particular case of Theorem 3.2.39 since, as we will show in the next chapter (Example 4.2.6), every such differentiation operator in $H_{bc}(E)$ is a convolution operator.

Example 3.2.41. If E' is separable with the approximation property then we can apply the previous theorem to the space $H_{Nb}(E)$ of nuclearly entire functions of bounded type. This answers a question of Aron and Markose in [AM04]. For E a dual Banach space and a slightly different definition of nuclear polynomials, Petersson obtained a stronger version of this result [Pet06].

Example 3.2.42. Let $\{\mathfrak{A}_k\}_k$ be a sequence of minimal ideals. If E' has the bounded approximation property, $\mathfrak{A}_k(E)'$ can be identified with $\mathfrak{B}_k(E') = \mathfrak{A}_k^*(E')$ (see Remark 3.2.32). Therefore, if E' is also separable and $\{\mathfrak{B}_k(E')\}_k$ is coherent, the convolution operators on $H_{b\mathfrak{A}}(E)$ are hypercyclic if they are not scalar multiples of the identity.

For example, we can take $\mathfrak{A}_k$ to be the minimal ideal associated to the tensor norm η_k [KR98, Car99]. In this case, $\{\mathfrak{B}_k\}_k$ is the coherent sequence of extendible polynomials.

In the next subsection, we present other examples in which the hypotheses of Theorem 3.2.39 are fulfilled. Namely, we consider the holomorphic functions of bounded type associated to the Schatten-von Neumann polynomials in the sense of Cobos, Kühn and Peetre [CKP92].

3.2.5 Schatten-von Neumann entire functions of bounded type

In this subsection we define of Schatten-von Neumann polynomials and holomorphic functions using the complex method of interpolation [Cal64, BL76]. We study some of their properties and show that the results from the previous subsection may be also applied in this case.

Suppose $\mathcal{H}$ is a separable Hilbert space. Let us first recall the definition of Hilbert-Schmidt n-homogeneous polynomials on $\mathcal{H}$, which will be denoted $\mathcal{S}_2^n(\mathcal{H})$. For finite type polynomials it is possible to define an inner product in the following way: if $y, z \in \mathcal{H}$, $\langle\, \langle \cdot, y\rangle^n, \langle \cdot, z\rangle^n \rangle = \langle z, y\rangle^n$. Then $\mathcal{S}_2^n(\mathcal{H})$ is the completion of the space of finite type polynomials $\mathcal{P}_f^n(\mathcal{H})$ with this inner product.

Note that if $\{e_i\}_i$ is an orthonormal basis of $\mathcal{H}$ and $P, Q \in \mathcal{S}_2^n(\mathcal{H})$, then

$$\langle P, Q\rangle = \sum_{i_1,\dots,i_n=1}^{\infty} \check{P}(e_{i_1},\dots,e_{i_n})\overline{\check{Q}(e_{i_1},\dots,e_{i_n})}.$$

Also note that the Borel transform is an isometric isomorphism between $(\mathcal{S}_2^n(\mathcal{H}))'$ and $\mathcal{S}_2^n(\mathcal{H}')$.

Cobos, Kühn and Peetre in [CKP92] defined the Schatten-von Neumann classes of multilinear functionals on $\mathcal{H}$. We adapt their definition to homogeneous polynomials on $\mathcal{H}$. To this end, throughout this section we will denote by $\mathcal{S}_1^n(\mathcal{H})$ and $\mathcal{S}_\infty^n(\mathcal{H})$ the spaces of n-homogeneous nuclear and approximable polynomials on $\mathcal{H}$, respectively. Since $\mathcal{H}'$ has the approximation property and the Radon-Nikodym property, the Borel transform is an isometric isomorphism between $(\mathcal{S}_1^n(\mathcal{H}))'$ and $\mathcal{P}^n(\mathcal{H}')$, and also between $(\mathcal{S}_\infty^n(\mathcal{H}))'$ and $\mathcal{S}_1^n(\mathcal{H}')$.

Following [CKP92], we define:

Definition 3.2.43. *The Schatten-von Neumann p-class of n-homogeneous polynomials on $\mathcal{H}$ is defined as*

$$\mathcal{S}_p^n(\mathcal{H}) = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\theta,$$

with $\frac{1}{p} = 1 - \theta$ and $0 < \theta < 1$. Here, $[\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\theta$ denotes the space obtained by complex interpolation from the pair $(\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H}))$, with parameter θ.

The following result, which is the polynomial version of [CKP92, Theorem 3.1] and can be proved analogously, shows that this definition is consistent with the definition of $\mathcal{S}_2^n(\mathcal{H})$.

Proposition 3.2.44. *We have the following isometric isomorphisms*

$$[\mathcal{S}_1^n(\mathcal{H}), \mathcal{P}^n(\mathcal{H})]_{1/2} \overset{1}{=} [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_{1/2} \overset{1}{=} \mathcal{S}_2^n(\mathcal{H}).$$

The above proposition may be obtained using the real method of interpolation, but only equivalent norms are achieved (see [CKP92]).

In the proof of [CKP92, Theorem 4.5], the reflexivity of $\mathcal{S}_p^n(\mathcal{H})$ is proven. In fact, this can be seen as a consequence of the following result:

Proposition 3.2.45. *If $1 < p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then the Borel transform is an isometric isomorphism between $(\mathcal{S}_p^n(\mathcal{H}))'$ and $\mathcal{S}_q^n(\mathcal{H}')$.*

Proof. We know that the statement holds when $p = 2$. Next, assume $1 < p < 2$.

By the Reiteration Theorem [BL76, 4.6.1] for the complex method,

$$\mathcal{S}_p^n(\mathcal{H}) = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\theta = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})]_\eta,$$

where $\theta = \frac{\eta}{2}$. Then $\mathcal{S}_p^n(\mathcal{H}) = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})]_{2\theta}$.

In the following two cases, the Borel transform is an isomorphism

$$\begin{array}{llll} B_n: & (\mathcal{S}_1^n(\mathcal{H}))' & \to & \mathcal{P}^n(\mathcal{H}') \qquad \text{and,} \\ B_n: & (\mathcal{S}_2^n(\mathcal{H}))' & \to & \mathcal{S}_2^n(\mathcal{H}'). \end{array}$$

Then $B_n : [(\mathcal{S}_1^n(\mathcal{H}))', (\mathcal{S}_2^n(\mathcal{H}))']_{2\theta} \to [\mathcal{P}^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')]_{2\theta}$ is an isomorphism.

Since $\mathcal{S}_2^n(\mathcal{H})$ is reflexive, and by a duality theorem [BL76, Corollary 4.5.2.], we have

$$[(\mathcal{S}_1^n(\mathcal{H}))', (\mathcal{S}_2^n(\mathcal{H}))']_{2\theta} = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_2^n(\mathcal{H})]'_{2\theta} = (\mathcal{S}_p^n(\mathcal{H}))'.$$

On the other hand,

$$[\mathcal{P}^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')]_{2\theta} = [\mathcal{S}_\infty^n(\mathcal{H}'), \mathcal{S}_2^n(\mathcal{H}')]_{2\theta} = [\mathcal{S}_1^n(\mathcal{H}'), \mathcal{S}_\infty^n(\mathcal{H}')]_\nu,$$

with $\nu = \frac{1}{2}2\theta + (1 - 2\theta) = 1 - \theta$ (the first equality follows from [BL76, Theorem 4.2.2.] and the last one from the Reiteration Theorem). Therefore, $B_n : (\mathcal{S}_p^n(\mathcal{H}))' \to [\mathcal{S}_1^n(\mathcal{H}'), \mathcal{S}_\infty^n(\mathcal{H}')]_\nu = \mathcal{S}_q^n(\mathcal{H}')$ is an isomorphism, with $\frac{1}{q} = 1 - \nu = \theta$, that is, $\frac{1}{q} = 1 - \frac{1}{p}$.

For the case $2 < p < \infty$, we have

$$\mathcal{S}_p^n(\mathcal{H}) = [\mathcal{S}_1^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\theta = [\mathcal{S}_2^n(\mathcal{H}), \mathcal{S}_\infty^n(\mathcal{H})]_\eta,$$

where $\eta = 2\theta - 1$. We proceed analogously to obtain the desired result. $\square$

Corollary 3.2.46. *For $1 < p < \infty$, the Schatten-von Neumann classes $\mathcal{S}_p^n(\mathcal{H})$ are reflexive.*

Since complex interpolation method is an exact interpolation functor of exponent θ, we have from Proposition 3.1.19:

Corollary 3.2.47. *For every $1 \leq p \leq \infty$, the sequence of Schatten-von Neumann p-classes of homogeneous polynomials $\{\mathcal{S}_p^k(\mathcal{H})\}_k$ is coherent.*

We denote by $H_{b\mathcal{S}_p}(\mathcal{H})$ be the holomorphic functions of bounded type of the Schatten-von Neumann p-class on $\mathcal{H}$. From Propositions 3.2.45 and 3.2.35 we have:

Corollary 3.2.48. *If $1 < p, q < \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then the Borel transform is a vector space isomorphism between $H_{b\mathcal{S}_p}(\mathcal{H})'$ and the q-Schatten functions of exponential type, $Exp_{\mathcal{S}_q}(\mathcal{H})$.*

By Proposition 3.2.45 we may apply Theorem 3.2.39 to the Schatten-von Neumann functions and thus we have:

Corollary 3.2.49. *Let $\mathcal{H}$ be separable Hilbert space. For $1 \leq p \leq \infty$, every convolution operator on $H_{b\mathcal{S}_p}(\mathcal{H})$ which is not a scalar multiple of the identity is hypercyclic.*

The case $p = 2$ of this result (Hilbert-Schmidt holomorphic functions of bounded type) was proved by Petersson in [Pet01].

Remark 3.2.50. We can also use the real interpolation method ([BL76, Chapter 3]) to define similar classes $\mathcal{S}_p$. Indeed this was also done in [CKP92], where the authors proved that the Hilbert-Schmidt trilinear forms are $(\mathcal{S}_1, \mathcal{S}_\infty)_{1/2,2}$ with equivalent norms.

Let E be any Banach space, we define $\tilde{\mathcal{S}}_p^k(E) := (\mathcal{S}_1^k(E), \mathcal{S}_\infty^k(E))_{\theta,2}$, with $\theta = 1 - \frac{1}{p}$ (note that in [CKP92, p.136], the class $\mathcal{S}_p$ on Hilbert spaces was defined using the real method by $(\mathcal{S}_1, \mathcal{S}_\infty)_{\theta,p}$), where S_1^k and S_∞^k denote the nuclear and approximable k-homogeneous polynomials respectively.

Proposition 3.2.51. *Let E be a Banach space whose dual is an Asplund space with the approximation property, $\frac{1}{p} + \frac{1}{q} = 1$. Then $\tilde{\mathcal{S}}_p^k(E)' = \tilde{\mathcal{S}}_q^k(E')$, with equivalent norms.*

Proof.

$$\begin{aligned}
\tilde{\mathcal{S}}_p^k(E)' &= (\mathcal{S}_1^k(E), \mathcal{S}_\infty^k(E))'_{\theta,2} \\
&\underset{(1)}{=} (\mathcal{S}_1^k(E)', \mathcal{S}_\infty^k(E)')_{\theta,2} \\
&= (\mathcal{P}^k(E'), \mathcal{P}_I^k(E'))_{\theta,2} \\
&\underset{(2)}{=} (\mathcal{S}_\infty^k(E'), \mathcal{P}_N^k(E'))_{\theta,2} \\
&\underset{(3)}{=} (\mathcal{P}_N^k(E'), \mathcal{S}_\infty^k(E'))_{1-\theta,2} = \tilde{\mathcal{S}}_q^k(E'),
\end{aligned}$$

where in (1) we used the duality theorem [BL76, Theorem 3.7.1], in (2) we used [BL76, Theorem 3.4.2 (d)] and the fact that for Asplund spaces nuclear and integral polynomials coincide [Ale85a, CD00], and in (3), [BL76, Theorem 3.4.1 (a)]. □

By Proposition 3.1.19 the sequence $\{\tilde{\mathcal{S}}_p^k\}_k$ is coherent, so we may define the $\tilde{\mathcal{S}}_p$-entire functions of bounded type, $H_{b\tilde{\mathcal{S}}_p}(E)$. Therefore we have that if E is a Banach space and E' is Asplund, separable and with the approximation property, every convolution operator not multiple of identity on $H_{b\tilde{\mathcal{S}}_p}(E)$ is hypercyclic.

3.2.6 Holomorphic mappings on open sets

In this subsection we define $\mathfrak{A}$-holomorphic functions on open subsets $U \subset E$ and more general on Riemann domains spread over E. The definition when U is a ball is almost immediate: we defined entire mappings as functions f such that $\lim \|\frac{d^k f(0)}{k!}\|_{\mathfrak{A}_k}^{\frac{1}{k}} = 0$; this may be interpreted as if the "$\mathfrak{A}$-radius of convergence" of f is ∞. Analogously, the $\mathfrak{A}$-holomorphic mappings of bounded type on a ball of radius r should have "$\mathfrak{A}$-radius of convergence" equal r.

Definition 3.2.52. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a coherent sequence of polynomial ideals; E, F Banach spaces and $x \in E$. Let $B_r(x) = B_E(x, r)$ be the open ball of radius r and center x in E. We define the space of $\mathfrak{A}$-holomorphic functions of bounded type on $B_r(x)$ by*

$$H_{b\mathfrak{A}}(B_r(x), F) = \left\{ f \in H(B_r(x), F) \ : \frac{d^k f(x)}{k!} \in \mathfrak{A}_k(E, F) \text{ and } \limsup_{k\to\infty} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k}^{1/k} \leq \frac{1}{r} \right\}.$$

We consider in $H_{b\mathfrak{A}}(B_r(x), F)$ the seminorms p_s, for $0 < s < r$, given by

$$p_s(f) = \sum_{k=0}^{\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k} s^k,$$

for all $f \in H_{b\mathfrak{A}}(B_r(x), F)$.

Proposition 3.2.53. *Let $\mathfrak{A}$ be a coherent sequence and $r > 0$. For every Banach spaces E and F and $x \in E$, $\Big(H_{b\mathfrak{A}}(B_r(x), F), \{p_s\}_{0<s<r}\Big)$ is a Fréchet space.*

Proof. Note that it suffices to consider the seminorms $\{p_{r(1-\frac{1}{n})}\}_{n\in\mathbb{N}}$, and therefore the space $\Big(H_{b\mathfrak{A}}(B_r(x), F), \{p_s\}_{0<s<r}\Big)$ is metrizable. The completeness may proved as in Proposition 3.2.2. $\square$

We also have results similar to those about Schauder decompositions for spaces of entire mappings given in Subsection 3.2.1.

Proposition 3.2.54. *Let $\mathfrak{A}$ be a coherent sequence and $r > 0$. Then $\{\mathfrak{A}_k(E, F)\}_k$ is a r-Schauder decomposition of $H_{b\mathfrak{A}}(B_r(x), F)$.*

Remark 3.2.55. By [GMR00, Remark 5], it is not possible to have a topological isomorphism between a Fréchet space with an R-Schauder decomposition ($0 < R < \infty$) and a Fréchet space with an ∞-Schauder decomposition. So, for every coherent sequence $\mathfrak{A}$ and every Banach spaces E and F, the spaces $H_{b\mathfrak{A}}(E, F)$ and $H_{b\mathfrak{A}}(B_r(x), F)$ are not isomorphic.

Also, analogously to Proposition 3.2.13, we have:

Proposition 3.2.56. *Let $\mathfrak{A}$ be a coherent sequence of polynomial ideals and let E and F be Banach spaces and $r > 0$. Then:*

(*a*) *$H_{b\mathfrak{A}}(B_r(x), F)$ is reflexive if and only if $\mathfrak{A}_k(E, F)$ is reflexive, for all k.*

(*b*) *If E is Asplund, $H_{Nb}(B_r(x), F)$ is topologically isomorphic to $H_{bI}(B_r(x), F)$.*

(*c*) *$H_{b\mathfrak{A}}(B_r(x), F)$ contains copy of c_0 if and only if there exists $k \in \mathbb{N}$ such that $\mathfrak{A}_k(E, F)$ contains copy of c_0.*

Observe that, even though $H_{b\mathfrak{A}}(E,F)$ and $H_{b\mathfrak{A}}(B_r(x),F)$ are never isomorphic, it follows from the previous proposition and Proposition 3.2.13 that they are both reflexive or none of them are. Also, both contain c_0 or no one does.

To finish this chapter we define $\mathfrak{A}$-holomorphic mappings of bounded type on general open subsets and on Riemann domains. This will be done in two steps. First we will define a space of holomorphic mappings on a Riemann domain X which resembles much the definition given in [DV04, Section 3] of the space $H_d(X)$ of holomorphic functions on X that are bounded on every ball which is X-bounded, but associated to a coherent sequence. Then the $\mathfrak{A}$-holomorphic mappings of bounded type on X will be asked to have some kind of "uniform boundedness" on X-bounded sets.

Let (X,p) be a Riemann domain over E and $x \in X$. A ball $B_r(x)$ is a subset of X such that $p|_{B_r(x)} : B_r(x) \to B_r(p(x))$ is an homeomorphism, and $d_X : X \to \mathbb{R}_{>0}$ is the function defined by: $d_X(x) = \sup\{r > 0 : \, B_r(x) \text{ exists}\}$.

Definition 3.2.57. *Let F be a Banach space. We will say that a mapping is in $H_{d\mathfrak{A}}(X,F)$ if it is $\mathfrak{A}$-holomorphic of bounded type on each ball in X, that is,*

$$H_{d\mathfrak{A}}(X,F) := \left\{ f \in H(X,F) \ : \textit{for every } x \in X, \ f \circ (p|_{B_s(x)})^{-1} \in H_{b\mathfrak{A}}(p(B_s(x)),F), \ \forall s < d_X(x) \right\}. \tag{3.4}$$

We define the seminorms $p_s^x(f)$ by

$$p_s^x(f) = \sum_{k=0}^{\infty} s^k \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k(E,F)},$$

for $0 < s < d_X(x)$, $x \in X$ and where $\frac{d^k f(x)}{k!} := \frac{d^k [f \circ (p|_{B_s(x)})^{-1}]}{k!}\big(p(x)\big)$. These seminorms define a topology on $H_{d\mathfrak{A}}(X)$ which is always complete (see Remark 3.2.61) but not necessarily a Fréchet topology unless E is separable. In that case we may copy the proof of [DV04, Proposition 3.2] to obtain:

Proposition 3.2.58. *Let $\mathfrak{A}$ be a coherent sequence and (X,p) be a connected Riemann domain over a separable Banach space E, then $H_{d\mathfrak{A}}(X,F)$ is a Fréchet space.*

The definition of $H_{d\mathfrak{A}}$ was inspired in [DV04], indeed, if $\mathfrak{A}$ is the sequence of all homogeneous polynomials and $F = \mathbb{C}$ then $H_{d\mathfrak{A}}(X) = H_d(X)$ defined in [DV04, Section 3]. It was shown there that $H_d = H_b$ for balls but they have shown an example of a bounded open subset U of ℓ_2 such that $H_b(U) \subsetneq H_d(U)$.

Definition 3.2.59. *We define the $\mathfrak{A}$-holomorphic mappings of bounded type on X as the mappings in $H_{d\mathfrak{A}}(X,F)$ such that, on each X-bounded open set A, the seminorms p_s^x (with $x \in A$ and $B_s(x) \subset A$) are uniformly bounded. That is, if we define $p_A(f) := \sup\{p_s^x(f) : \, B_s(x) \subset A\}$, then*

$$H_{b\mathfrak{A}}(X,F) = \{ f \in H_{d\mathfrak{A}}(X,F) \ : \textit{for every } x \in A \subset\subset X, \ A \textit{ open}, p_A(f) < \infty \}. \tag{3.5}$$

For $\mathfrak{A}$ the sequence of all homogeneous polynomials, $H_{b\mathfrak{A}}(X,F) = H_b(X,F)$.

Proposition 3.2.60. *The seminorms $\{p_A : \, A \subset\subset X, \ A \text{ open}\}$ define a Fréchet space topology on $H_{b\mathfrak{A}}(X,F)$.*

Proof. It is clear that the topology may be described with the countable set of seminorms $\{p_{X_n}\}_{n\in\mathbb{N}}$, where $X_n = \{x \in X : \|p(x)\| \leq n \text{ and } d_X(x) \geq \frac{1}{n}\}$, so we only need to prove completeness. Let (f_k) be a Cauchy sequence in $H_{b\mathfrak{A}}(X,F)$, then it is a Cauchy sequence in $H_b(X,F)$, so there exists a function $f \in H_b(X,F)$ which is limit (uniformly in X-bounded sets) of the f_k's. Let $A \subset\subset X$, $x \in A$ and $r < d_X(x)$. Then $(f_k \circ (p|_{B_r(x)})^{-1})$ is a Cauchy sequence in $H_{b\mathfrak{A}}(B_r(p(x)),F)$ which converge to $f \circ (p|_{B_r(x)})^{-1})$. Since $H_{b\mathfrak{A}}(B_r(p(x)),F)$ is complete we have that $f \circ (p|_{B_r(x)})^{-1})$ belongs to $H_{b\mathfrak{A}}(B_r(p(x)),F)$. Moreover, $p_s^x(f) \leq \sup_k p_s^x(f_k)$ for every $s < r$. Note also that p_A is bounded in the Cauchy sequence (f_k), therefore

$$p_A(f) = \sup\{p_s^x(f) :\ B_s(x) \subset A\} \leq \sup_k p_A(f_k) < \infty.$$

$\square$

Remark 3.2.61. Note that we have also proved that $H_{d\mathfrak{A}}(X,F)$ is complete.

Chapter 4

Multiplicative sequences and algebras of entire functions of bounded type

We define the concept of multiplicative sequence $\mathfrak{A}$ of scalar polynomial ideals. This allows us to associate an algebra of entire functions of bounded type $H_{b\mathfrak{A}}(E)$ to a coherent and multiplicative sequence of polynomial ideals. We study convolution operators on $H_{b\mathfrak{A}}(E)$. We prove that, under very natural conditions satisfied by many examples of sequences, the spectrum of the associated algebra "behaves" like the classical case of $M_b(E)$ (the spectrum of the algebra of all entire functions of bounded type, $H_b(E)$). More precisely, we prove that $M_{b\mathfrak{A}}(E)$ can be endowed with a structure of Riemann domain over E'' and that the extension of each $f \in H_{b\mathfrak{A}}(E)$ to the spectrum is an $\mathfrak{A}$-holomorphic function of bounded type in each connected component. We also prove a Banach-Stone type theorem for these algebras of holomorphic functions.

We also investigate how to define algebras of holomorphic functions associated to sequences of polynomial ideals on more general open sets. In [CDM07, CDM], we can find most of the material appearing in this chapter.

4.1 Multiplicative sequences

In this chapter we will study algebras of holomorphic functions of bounded type associated to sequences of polynomial ideals. Let $\mathfrak{A} = \{\mathfrak{A}_k\}$ be a coherent sequence at E. We will write $\mathfrak{A}(E)$ to denote the sequence $\{\mathfrak{A}_k(E)\}$. Condition (ii) in the definition of coherence states that the product of a polynomial in $\mathfrak{A}(E)$ by a linear functional remains in $\mathfrak{A}(E)$. But if we take two polynomials in $\mathfrak{A}(E)$, is their product in $\mathfrak{A}(E)$? As the following example shows, this is not necessarily the case.

Example 4.1.1. The construction in Section 3.1.1 can be easily adapted to obtain a coherent sequence $\{\mathfrak{A}_n\}_n$ with $\mathfrak{A}_1 = \mathcal{L}$, $\mathfrak{A}_2 = \mathcal{P}^2$, $\mathfrak{A}_3 = \mathcal{P}^3$ and $\mathfrak{A}_4 = \mathcal{P}^4_{wsc0}$ (the 4-homogeneous polynomials that are weakly sequentially continuous at 0). To see that the sequence is not multiplicative consider, for instance, $P \in \mathcal{P}^2(\ell_2)$ given by $P(x) = \sum_n x_n^2$. Then $P \in \mathfrak{A}_2(\ell_2)$ but $P^2 \notin \mathfrak{A}_4(\ell_2)$.

Thus, in order to obtain that $H_{b\mathfrak{A}}(E)$ is an algebra we introduce the following:

Definition 4.1.2. *A coherent sequence at E, $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ is **multiplicative** (at E) if there exists $M \geq 1$ such that for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have $PQ \in \mathfrak{A}_{k+l}(E)$ and*

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq M^{k+l} \|P\|_{\mathfrak{A}_k(E)} \|Q\|_{\mathfrak{A}_l(E)}.$$

Remark 4.1.3. *Example 4.1.1 shows that not every coherent sequence is multiplicative.*

We will show now that $\{\mathfrak{A}_k(E)\}_k$ is multiplicative, then $H_{b\mathfrak{A}}(E)$ is a B_0-algebra, that is a complete metrizable topological algebra such that the topology is given by means of an increasing sequence $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ of seminorms satisfying that $\|xy\|_j \leq C_i\|x\|_{j+1}\|y\|_{j+1}$ for every x, y in the algebra and every $j \geq 1$, where C_i are positive constants.[1] In Section 4.4 we will prove that in many cases (like the nuclear or integral) $H_{b\mathfrak{A}}(E)$ is a locally m-convex algebra.

Lemma 4.1.4. *Let $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ be a multiplicative sequence of normed ideals. If $f, g \in H_{b\mathfrak{A}}(E)$ then $fg \in H_{b\mathfrak{A}}(E)$. Moreover, $H_{b\mathfrak{A}}(E)$ is a B_0-algebra.*

Proof. We already know that $H_{b\mathfrak{A}}(E)$ is a Fréchet space. Moreover, the topology may be given by the sequence of seminorms $\{p_{M^n}\}_{n\in\mathbb{N}}$ (see Section 3.2). Take $f, g \in H_{b\mathfrak{A}}(E)$ with Taylor expansions $f = \sum_{k=0}^{\infty} P_k$ and $g = \sum_{k=0}^{\infty} Q_k$, where $P_k, Q_k \in \mathfrak{A}_k(E)$. Then $\dfrac{d^k(fg)(0)}{k!} = \sum_{j=0}^{k} P_j Q_{k-j}$ belongs to $\mathfrak{A}_k(E)$, since $\mathfrak{A}(E)$ is multiplicative. Take $r = M^n$, with $n \in \mathbb{N}$ then

$$\begin{aligned}
\sum_k r^k \Big\| \frac{d^k(fg)(0)}{k!} \Big\|_{\mathfrak{A}_k(E)} &= \sum_k r^k \Big\| \sum_{j=0}^{k} P_j Q_{k-j} \Big\|_{\mathfrak{A}_k(E)} \leq \sum_k r^k M^k \sum_{j=0}^{k} \|P_j\|_{\mathfrak{A}_j(E)} \|Q_{k-j}\|_{\mathfrak{A}_{k-j}(E)} \\
&= \sum_{j=0}^{\infty} (rM)^j \|P_j\|_{\mathfrak{A}_j(E)} \sum_{k=j}^{\infty} (rM)^{k-j} \|Q_{k-j}\|_{\mathfrak{A}_{k-j}(E)} = p_{rM}(f) p_{rM}(g) < \infty.
\end{aligned}$$

Therefore $fg \in H_{b\mathfrak{A}}(E)$ and $p_r(fg) \leq p_{rM}(f)p_{rM}(g)$. Therefore $H_{b\mathfrak{A}}(E)$ is a B_0-algebra. $\square$

Below we give some examples of sequences of polynomial ideals which are multiplicative at any Banach space. The coherence was already shown in the previous chapter.

Example 4.1.5. (a) If $\mathfrak{A}_k$ is the ideal of all k-homogeneous (or of compact, weakly compact, approximable, extendible) polynomials then it easy to see that $\{\mathfrak{A}_k(E)\}_k$ is a multiplicative sequence with constant $M = 1$.

(b) If $\mathfrak{A}_k = \mathcal{P}_N^k$ is the ideal of all k-homogeneous nuclear polynomials then $\{\mathfrak{A}_k(E)\}_k$ is a multiplicative sequence. This was proved by Dineen [Din71a, Lemma 15] for separable Hilbert spaces. The general case may be deduced as a consequence of the following example and Corollary 4.1.14 (see Example 4.4.8).

(c) If $\mathfrak{A}_k$ is the ideal of all k-homogeneous integral polynomials then $\{\mathfrak{A}_k(E)\}_k$ is a multiplicative sequence. Indeed, for $P \in \mathcal{P}_I(^kE)$, $Q \in \mathcal{P}_I(^lE)$, let us prove that PQ is a continuous linear functional on the k-fold symmetric tensor product of E with the injective symmetric norm (ε_k^s). Take $\psi = \sum_i x_i^{k+l} \in \bigotimes_{\varepsilon_{k+l}^s}^{k+l,s} E'$, then

$$\begin{aligned}
\big|\langle PQ, \psi\rangle\big| &= \big|\sum_i P(x_i)Q(x_i)\big| = \big|P\big(\sum_i x_i^k Q(x_i)\big)\big| \leq \|P\|_I \sup_{\gamma\in B_{E'}} \big|\sum_i \gamma(x_i)^k Q(x_i)\big| \\
&= \|P\|_I \sup_{\gamma\in B_{E'}} \Big|Q\Big(\sum_i \gamma(x_i)^k x_i^l\Big)\Big| \leq \|P\|_I \|Q\|_I \sup_{\gamma\in B_{E'}} \sup_{\varphi\in B_{E'}} \big|\sum_i \gamma(x_i)^k \varphi(x_i)^l\big|.
\end{aligned}$$

[1] It is possible to make $C_i = 1$ for all i.

Let $R \in \mathcal{P}(^{k+l}E')$, $R(\gamma) := \sum_i \gamma(x_i)^{k+l}$ for $\gamma \in E'$. Then by [Har97, Corollary 4], if we take $\gamma, \varphi \in S_{E'}$, we obtain

$$|\overset{\vee}{R}(\underbrace{\gamma,\dots,\gamma}_{k},\underbrace{\varphi,\dots,\varphi}_{l})| = \big|\sum_i \gamma(x_i)^k\varphi(x_i)^l\big| \leq \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\|R\|$$
$$= \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\varepsilon^s_{k+l}\big(\sum_i x_i^{k+l}\big).$$

Therefore,

$$\big|\langle PQ,\psi\rangle\big| \leq \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\|P\|_I\|Q\|_I\varepsilon^s_{k+l}(\psi),$$

and so PQ is integral with $\|PQ\|_I \leq \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\|P\|_I\|Q\|_I \leq e^{k+l}\|P\|_I\|Q\|_I$.

(d) If $\mathfrak{A}_k$ is the ideal of all k-homogeneous multiple r-summing polynomials then $\{\mathfrak{A}_k(E)\}_k$ is a multiplicative sequence with constant $M = 1$. The proof is similar Example 3.1.9, but the notation is more messy.

Let $P \in \mathcal{M}_r^k(E), Q \in \mathcal{M}_r^l(E)$, then

$$(PQ)^\vee(x_1,\dots,x_{k+l}) = \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\overset{\vee}{P}(x_1,\overset{s_1\dots s_l}{\dots},x_{k+l})\overset{\vee}{Q}(x_{s_1},\dots,x_{s_l})$$

where $\overset{\vee}{P}(x_1,\overset{s_1\dots s_l}{\dots},x_{k+l})$ means that coordinates $x_{s_1},\dots,x_{s_l}$ are omitted.

Take $(x_j^{i_j})_{j=1}^{m_j} \subset E$, for $j = 1,\dots,k+l$, such that $w_r((x_j^{i_j})) = 1$. Then, using the triangle inequality for the ℓ_r-norm,

$$\left(\sum_{i_1,\dots,i_{k+l}=1}^{m_1,\dots,m_{k+l}}\|(PQ)^\vee(x_1^{i_1},\dots,x_{k+l}^{i_{k+l}})\|^r\right)^{\frac{1}{r}} \leq$$
$$\leq \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\left(\sum_{i_1,\dots,i_{k+l}=1}^{m_1,\dots,m_{k+l}}\left|\overset{\vee}{P}(x_1,\overset{s_1\dots s_l}{\dots},x_{k+l})\right|^r\left|\overset{\vee}{Q}(x_{s_1},\dots,x_{s_l})\right|^r\right)^{1/r}$$
$$\leq \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\left(\sum_{i_{s_1},\dots,i_{s_l}=1}^{m_{s_1},\dots,m_{s_l}}\left|\overset{\vee}{Q}(x_{s_1},\dots,x_{s_l})\right|^r\|P\|^r_{\mathcal{M}_r^k}\right)^{1/r}$$
$$\leq \frac{k!}{(k+l)!}\sum_{\substack{s_1,\dots,s_l=1\\ s_1\neq\dots\neq s_l}}^{k+l}\|P\|_{\mathcal{M}_r^k}\|Q\|_{\mathcal{M}_r^l}.$$

Hence, PQ is multiple r-summing with $\|PQ\|_{\mathcal{M}_r^{k+l}} \leq \|P\|_{\mathcal{M}_r^k}\|Q\|_{\mathcal{M}_r^l}$.

Also, given sequences which are multiplicative we can obtain new multiplicative sequences from them. For example, interpolation of multiplicative sequences is multiplicative. In this way we can obtain many other examples of multiplicative sequences. Here we will use the complex interpolation method, but any other method with a nice bilinear interpolation theorem would work.

Proposition 4.1.6. *Let E be Banach space and let $\{\mathfrak{A}_k^0(E)\}_k$, $\{\mathfrak{A}_k^1(E)\}_k$ be coherent multiplicative sequences (with constants M_0 and M_1, respectively). Then, the sequence $\{\mathfrak{A}_k^\theta(E)\}_k$ is multiplicative, where $\mathfrak{A}_k^\theta(E) = [\mathfrak{A}_k^0(E), \mathfrak{A}_k^1(E)]_\theta$, for every $0 < \theta < 1$ (with constant $M_0^{1-\theta} M_1^\theta$).*

Proof. We already know that interpolation of coherent sequences is coherent (Proposition 3.1.19). Since $\{\mathfrak{A}_k^j(E)\}_k$ is multiplicative, for $j = 0, 1$, we can define a continuous bilinear mapping

$$\begin{array}{rccc} \Phi_{k,l}^j : & \mathfrak{A}_k^j(E) \times \mathfrak{A}_l^j(E) & \to & \mathfrak{A}_{k+l}^j(E) \\ & (P, Q) & \mapsto & PQ. \end{array}$$

It follows that $\|\Phi_{k,l}^j\| \leq M_j^{k+l}$. Then, by the Multilinear Interpolation Theorem [BL76, Theorem 4.4.1.], $(P, Q) \mapsto PQ$ defines a mapping

$$\Phi_{k,l}^\theta : \quad \mathfrak{A}_k^\theta(E) \times \mathfrak{A}_l^\theta(E) \quad \to \quad \mathfrak{A}_{k+l}^\theta(E),$$

which is continuous and has norm less than or equal to $\left(M_0^{1-\theta} M_1^\theta\right)^{k+l}$. That is, if $P \in \mathfrak{A}_k^\theta(E)$, $Q \in \mathfrak{A}_l^\theta(E)$, then $PQ \in \mathfrak{A}_{k+l}^\theta(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}^\theta(E)} \leq \left(M_0^{1-\theta} M_1^\theta\right)^{k+l} \|P\|_{\mathfrak{A}_k^\theta(E)} \|Q\|_{\mathfrak{A}_l^\theta(E)}.$$

□

Example 4.1.7. The sequence $\{\tilde{\mathcal{S}}_p^k\}_k$ of Schatten-von Neumann polynomials on a Hilbert space $\mathcal{H}$ (defined in Subsection 3.2.5) is multiplicative. For example, for the Hilbert-Schmidt polynomials ($p = 2$) we have that if $P \in \mathcal{S}_2^k(\mathcal{H})$ and $Q \in \mathcal{S}_2^l(\mathcal{H})$ then $PQ \in \mathcal{S}_2^{k+l}(\mathcal{H})$ and $\|PQ\|_{\mathcal{S}_2^{k+l}} \leq \sqrt{e}^{k+l} \|P\|_{\mathcal{S}_2^k} \|Q\|_{\mathcal{S}_2^l}$. In [Pet01, Lemma 2.1] it had been proved that $\|PQ\|_{\mathcal{S}_2^{k+l}} \leq 2^{k+l} \|P\|_{\mathcal{S}_2^k} \|Q\|_{\mathcal{S}_2^l}$.

We can also obtain multiplicative sequences using the composition of ideals, as the next two propositions show.

Proposition 4.1.8. *Let $\{\mathfrak{A}_k\}_k$ be a sequence of polynomial ideals and $\mathfrak{C}$ a closed ideal of operators. Suppose that for each Banach space E, each time that $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$ then $PQ \in \mathfrak{A}_{k+l}(E)$ and*

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq c \|P\|_{\mathfrak{A}_k(E)} \|Q\|_{\mathfrak{A}_l(E)}.$$

Then, the sequence $\{\mathfrak{A}_k \circ \mathfrak{C}\}_k$ has the same property.

Proof. Take $P \in \mathfrak{A}_k \circ \mathfrak{C}(E)$ and $Q \in \mathfrak{A}_l \circ \mathfrak{C}(E)$ and write them as $P = \tilde{P} \circ S$ and $Q = \tilde{Q} \circ T$, with $S \in \mathfrak{C}(E, E_1)$, $T \in \mathfrak{C}(E, E_2)$, $\|S\|_{\mathfrak{C}(E,E_1)} = \|T\|_{\mathfrak{C}(E,E_2)} = 1$, $\tilde{P} \in \mathfrak{A}_k(E_1)$ and $\tilde{Q} \in \mathfrak{A}_l(E_2)$. We consider the product space $E_1 \times E_2$ with the supremum norm and define $\tilde{S} : E \to E_1 \times E_2$ and $\tilde{T} : E \to E_1 \times E_2$ by $\tilde{S}(x) = (S(x), 0)$ and $\tilde{T}(x) = (0, T(x))$. Clearly, $\tilde{S}$ and $\tilde{T}$ belong to $\mathfrak{C}(E, E_1 \times E_2)$ and so does $\tilde{S} + \tilde{T}$. Moreover, the norm of $\tilde{S} + \tilde{T}$ in $\mathfrak{C}$ is the maximum of those of S and T, thus $\|\tilde{S} + \tilde{T}\|_{\mathfrak{C}(E, E_1 \times E_2)} = 1$.

On the other hand, in a similar way we can see that $R : E_1 \times E_2 \to \mathbb{K}$ given by $R(y_1, y_2) = \tilde{P}(y_1)\tilde{Q}(y_2)$ belongs to $\mathfrak{A}_{k+l}(E_1 \times E_2)$, and $\|R\|_{\mathfrak{A}_{k+l}(E_1 \times E_2)} \leq c \|\tilde{P}\|_{\mathfrak{A}_k(E_1)} \|\tilde{Q}\|_{\mathfrak{A}_l(E_2)}$. Since $PQ = R \circ (\tilde{S} + \tilde{T})$, we have that PQ belongs to $\mathfrak{A}_{k+l} \circ \mathfrak{C}$. Moreover,

$$\|PQ\|_{\mathfrak{A}_{k+l} \circ \mathfrak{C}(E)} \leq \|R\|_{\mathfrak{A}_{k+l}(E_1 \times E_2)} \|\tilde{S} + \tilde{T}\|_{\mathfrak{C}(E, E_1 \times E_2)}^{k+l} \leq c \|\tilde{P}\|_{\mathfrak{A}_k(E_1)} \|\tilde{Q}\|_{\mathfrak{A}_l(E_2)}.$$

Considering all the possible factorizations of P and Q (with operators of norm 1) we obtain the desired norm estimate. □

Corollary 4.1.9. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a multiplicative sequence and $\mathfrak{C}$ a closed ideal of operators. Then $\{\mathfrak{A}_k \circ \mathfrak{C}\}_k$ is a multiplicative sequence.*

Proof. Just combine Propositions 4.1.8 and 3.1.2. □

Proposition 4.1.10. *Let $\{\mathfrak{A}_k\}_k$ be a sequence of closed polynomial ideals and $\mathfrak{C}$ a normed ideal of operators. Suppose that for each Banach space E, each time that $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$ then $PQ \in \mathfrak{A}_{k+l}(E)$. Then, the sequence $\{\mathfrak{A}_k \circ \mathfrak{C}\}_k$ is multiplicative with constant $M = 1$.*

Proof. Take $P \in \mathfrak{A}_k \circ \mathfrak{C}(E)$ and $Q \in \mathfrak{A}_l \circ \mathfrak{C}(E)$ and write them as $P = \tilde{P} \circ S$ and $Q = \tilde{Q} \circ T$, with $S \in \mathfrak{C}(E, E_1)$, $T \in \mathfrak{C}(E, E_2)$, $\|S\|_{\mathfrak{C}(E,E_1)} = \|T\|_{\mathfrak{C}(E,E_2)} = 1$, $\tilde{P} \in \mathfrak{A}_k(E_1)$ and $\tilde{Q} \in \mathfrak{A}_l(E_2)$. As in the previous proposition, we consider the product space $E_1 \times E_2$, but with the ℓ_1 norm and define $\tilde{S} : E \to E_1 \times E_2$ and $\tilde{T} : E \to E_1 \times E_2$ by $\tilde{S}(x) = (S(x), 0)$ and $\tilde{T}(x) = (0, T(x))$. Clearly, $\tilde{S}$ and $\tilde{T}$ belong to $\mathcal{C}(E, E_1 \times E_2)$ and so does $\tilde{S} + \tilde{T}$. Moreover, $\|\tilde{S} + \tilde{T}\|_{\mathcal{C}(E,E_1\times E_2)} = 2$.

If $R : E_1 \times E_2 \to \mathbb{K}$ is given by $R(y_1, y_2) = \tilde{P}(y_1)\tilde{Q}(y_2)$, then it belongs to $\mathfrak{A}_{k+l}(E_1 \times E_2)$, and

$$\begin{aligned}\|R\|_{\mathfrak{A}_{k+l}(E_1\times E_2)} &= \sup_{(x_1,x_2)\in E_1\times E_2} \frac{|\tilde{P}(x_1)\tilde{Q}(x_2)|}{\|(x_1,x_2)\|_{E_1\times E_2}^{k+l}} \\ &\leq \sup_{\|x_1\|_{E_1}=\|x_2\|_{E_2}=1} \frac{|\tilde{P}(x_1)\tilde{Q}(x_2)|}{(\|x_1\|_{E_1} + \|x_2\|_{E_2})^{k+l}} = \frac{\|\tilde{P}\|_{\mathfrak{A}_k(E_1)}\|\tilde{Q}\|_{\mathfrak{A}_l(E_2)}}{2^{k+l}}.\end{aligned}$$

Since $PQ = R \circ (\tilde{S} + \tilde{T})$, we have that PQ belongs to $\mathfrak{A}_{k+l} \circ \mathfrak{C}$. Moreover,

$$\|PQ\|_{\mathfrak{A}_{k+l}\circ\mathfrak{C}(E)} \leq \|R\|_{\mathfrak{A}_{k+l}(E_1\times E_2)}\|\tilde{S} + \tilde{T}\|_{\mathfrak{C}(E,E_1\times E_2)}^{k+l} \leq \|\tilde{P}\|_{\mathfrak{A}_k(E_1)}\|\tilde{Q}\|_{\mathfrak{A}_l(E_2)}.$$

Considering all the possible factorizations of P and Q (with operators of norm 1) we obtain the desired norm estimate. □

Example 4.1.11. *The sequence $\{\mathcal{P}_\infty^k\}_k$ of ∞-factorable polynomials (also of strongly ∞-factorable or ∞-compact polynomials) is multiplicative with constant $M = 1$.*

Suppose we have a sequence of ideals which is related to tensor norms. The multiplicativity of the sequence then translates in properties of the tensor norms, depending on how the ideals relate to them. To illustrate this we state the following proposition, which is a generalization of Lemma 3.1.28:

Proposition 4.1.12. *For each k, let α_k be a finitely generated k-fold symmetric tensor norm α_k. Consider $\mathfrak{A}_k^{max}$ and $\mathfrak{A}_k^{min}$ maximal and minimal ideals associated to α_k. Then the following are equivalent.*

(i) For every Banach space E, if $P \in \mathfrak{A}_k^{max}(E)$ and $Q \in \mathfrak{A}_l^{max}(E)$ then $PQ \in \mathfrak{A}_{k+l}^{max}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}^{max}(E)} \leq c_{k,l}\|P\|_{\mathfrak{A}_k^{max}(E)}\|Q\|_{\mathfrak{A}_l^{max}(E)}.$$

(ii) For every Banach space E, if $P \in \mathfrak{A}_k^{min}(E)$ and $Q \in \mathfrak{A}_l^{min}(E)$ then $PQ \in \mathfrak{A}_{k+l}^{min}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}^{min}(E)} \leq c_{k,l}\|P\|_{\mathfrak{A}_k^{min}(E)}\|Q\|_{\mathfrak{A}_l^{min}(E)}.$$

(iii) For every Banach space E, if $s \in \bigotimes_{\alpha_k}^{k,s} E'$ and $t \in \bigotimes_{\alpha_l}^{l,s} E'$, then

$$\alpha_{k+l}(\sigma(s \otimes t)) \leq c_{k,l}\alpha_k(s)\alpha_l(t).$$

Proof. The three statements are clearly equivalent if E is a finite dimensional Banach space. Since α_k is finitely generated, (i) is implied by (ii) or by (iii) for all Banach spaces.

We now prove that (i) implies (iii). Note that (iii) is equivalent to prove that the bilinear map $\phi_E : \left(\bigotimes_{\alpha_k}^{k,s} E' \times \bigotimes_{\alpha_l}^{l,s} E', \|\cdot\|_\infty\right) \to \bigotimes_{\alpha_{k+l}}^{k+l,s} E'$, $\phi_E(s,t) = \sigma(s \otimes t)$ is continuous of norm $\leq c_{k,l}$ for every Banach space E. If (i) is true then ϕ_S is continuous (with norm $\leq c_{k,l}$) for every finite dimensional Banach space S. Let M, N be two finite dimensional subspaces of E' such that $s \in \bigotimes^{k,s} M$ and $t \in \bigotimes^{l,s} N$. Then

$$\begin{aligned} \alpha_{k+l}\big(\sigma(s\otimes t), \bigotimes^{k+l,s} M+N\big) &\leq c_{k,l} \max\{\alpha_k\big(s, \bigotimes^{k,s} M+N\big), \alpha_l\big(t, \bigotimes^{l,s} M+N\big)\} \\ &\leq c_{k,l} \max\{\alpha_k\big(s, \bigotimes^{k,s} M\big), \alpha_l\big(t, \bigotimes^{l,s} N\big)\}, \end{aligned}$$

where the second inequality is true by the metric mapping property. Taking the infimum over M and N we obtain that $\|\phi_E\| \leq c_{k,l}$ and thus we have (iii).

To see that (i) implies (ii), just note that $\mathfrak{A}_k^{min} = \mathfrak{A}_k^{max} \circ \overline{\mathcal{F}}$ and use Proposition 4.1.8. □

Remark 4.1.13. (a) Note that in the proof of the previous proposition, we have shown that if any of the three conditions are fulfilled for arbitrary finite dimensional Banach spaces, then every Banach space satisfy the three of them.

(b) Condition (iii) is one of the inequalities satisfied by a "family of complemented symmetric seminorms", defined by C. Boyd and S. Lassalle in [BL08].

Corollary 4.1.14. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a multiplicative sequence. Then $\{\mathfrak{A}_k^{min}\}_k$ and $\{\mathfrak{A}_k^{max}\}_k$ are multiplicative sequences.*

To finish this section we present another family of examples of multiplicative sequences: the natural sequences of polynomial ideals as defined in [CGb] are multiplicative (see Subsection 3.1.5, where we have recalled their definition).

For the multiplicativity of the polynomial ideals associated to the natural symmetric tensor norms we need the following:

Lemma 4.1.15. *For each k, let α_k be a finitely generated k-fold symmetric tensor and suppose $\alpha_{k+l}(\sigma(s\otimes t)) \leq c_{k,l}\alpha_k(s)\alpha_l(t)$ for every $s \in \bigotimes_{\alpha_k}^{k,s} E'$ and $t \in \bigotimes_{\alpha_l}^{l,s} E'$. Then the same inequality holds for the sequences $\{/\alpha_k\backslash\}_k$ and $\{\backslash\alpha_k/\}_k$.*

Proof. For all k, we denote

$$i_k = \otimes^k i : \big(\otimes^{k,s} E', /\alpha_k\backslash\big) \overset{1}{\hookrightarrow} \big(\otimes^{k,s} \ell_\infty(B_{E''}), \alpha_k\big),$$

then

$$\begin{aligned} /\alpha_{k+l}\backslash(\sigma(s\otimes t)) &= \alpha_{k+l}\Big(i_{k+l}\big(\sigma(s\otimes t)\big)\Big) = \alpha_{k+l}\Big(\sigma\big(i_k(s)\otimes i_l(t)\big)\Big) \\ &\leq c_{k,l}\alpha_k(i_k(s))\alpha_l(i_l(s)) = c_{k,l}/\alpha_k\backslash(s)/\alpha_l\backslash(t). \end{aligned}$$

On the other hand, if $\mathfrak{A}_k^{max}$ is the maximal ideal associated to α_k, and $\mathfrak{B}_k$ is the maximal polynomial ideal to $\backslash\alpha_k/$ then (see the proof of Lemma 3.1.34)

$$\mathfrak{B}_k(E) = \{P \in \mathcal{P}^k(E) : P \text{ extends to a polynomial } \tilde{P} \in \mathfrak{A}_k^{max}(\ell_\infty(B_{E'}))\},$$

and the norm of P in $\mathfrak{B}_k$ is given by the infimum of the $\mathfrak{A}_k^{max}$-norms of these extensions. Since $\mathfrak{A}_k^{max}$ satisfies condition (i) of Proposition 4.1.12, it is easy to see that the product of two polynomials in $\mathfrak{B}$ belongs to $\mathfrak{B}$ with the same inequality. Using Proposition 4.1.12 again, we obtain the desired result for $\{\backslash\alpha_k/\}_k$. □

In Theorem 3.1.36 we proved that the sequences of maximal and minimal polynomial ideals associated to natural sequences of symmetric tensor norms are coherent. Since moreover π_k and ε_k are multiplicative, we can use the previous lemma and Proposition 4.1.12 to show that:

Theorem 4.1.16. *Let $\{\alpha_k\}_k$ be any of the natural sequences of symmetric tensor norms. Then the sequences $\{\mathfrak{A}_k^{max}\}_k$ and $\{\mathfrak{A}_k^{min}\}_k$ of maximal and minimal ideals associated to $\{\alpha_k\}_k$ are multiplicative.*

To end this section we now show that there is some kind of duality between the properties of multiplicativity and weakly differentiability of a sequence defined in Section 3.2.2.

Proposition 4.1.17. *(i) Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a weakly differentiable sequence. Then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_k$ is multiplicative.*

(ii) Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a multiplicative sequence. Then the sequence of adjoint ideals $\{\mathfrak{A}_k^*\}_k$ is weakly differentiable.

In both cases the constants of multiplicativity and weakly differentiability are the same.

Proof. (i) From 3.1.30 we know that $\{\mathfrak{A}_k^*\}_k$ is coherent. By Remark 4.1.13, it suffices to check that $\|PQ\|_{\mathfrak{A}_{k+l}^*(M)} \leq K^{k+l}\|P\|_{\mathfrak{A}_k^*(M)}\|Q\|_{\mathfrak{A}_l^*(M)}$ for any finite dimensional Banach space M. Since M is finite dimensional, $\mathfrak{A}_k^*(M)$ is just $\mathfrak{A}_k(M')'$. Take $P \in \mathfrak{A}_k^*(M) = \mathfrak{A}_k(M')'$ and $Q \in \mathfrak{A}_l^*(M) = \mathfrak{A}_l(M')'$. For $\Psi = \sum_j x_j^{k+l} \in \mathfrak{A}_{k+l}(M')$, we have

$$\langle PQ, \Psi\rangle = \sum_j P(x_j)Q(x_j) = \langle Q, \sum_j \langle P, x_j^k\rangle x_j^l\rangle = \langle Q, \gamma \mapsto \sum_j \langle P, x_j^k\rangle x_j(\gamma)^l\rangle = \langle Q, \gamma \mapsto P(\Psi_{\gamma^l})\rangle.$$

Thus, since $\{\mathfrak{A}_k\}_k$ is weakly differentiable,

$$|\langle PQ, \Psi\rangle| \leq \|Q\|_{\mathfrak{A}_l^*(M)} \ \|\gamma \mapsto P(\Psi_{\gamma^l})\|_{\mathfrak{A}_l(M)} \leq \|Q\|_{\mathfrak{A}_l^*(M)} K^{k+l}\|P\|_{\mathfrak{A}_k^*(M)}\|\Psi\|_{\mathfrak{A}_{k+l}(M')},$$

which implies that $\|PQ\|_{\mathfrak{A}_{k+l}^*(M)} \leq K^{k+l}\|P\|_{\mathfrak{A}_k^*(M)}\|Q\|_{\mathfrak{A}_l^*(M)}$.

(ii) For each k, let α_k be the finitely generated symmetric tensor norm associated to $\mathfrak{A}_k$, so that for every E, $\mathfrak{A}_k^*(E) = \left(\bigotimes_{\alpha_k}^{k,s} E\right)'$. By Proposition 4.1.12 and Remark 4.1.13 (a), the multiplicativity of $\mathfrak{A}$ with constant M implies that $\alpha_k(\sigma(s \otimes t)) \leq M^k\alpha_l(s)\alpha_{k-l}(t)$, for every $s \in \bigotimes_{\alpha_l}^{l,s} E$ and $t \in \bigotimes_{\alpha_{k-l}}^{k-l,s} E$.

Take $P \in \mathfrak{A}_k^*(E)$ and $\varphi \in \mathfrak{A}_{k-l}^*(E)'$, with $\|\varphi\| = 1$. Define $Q(x) = \varphi(P_{x^l})$. Note that Q is a well defined l-homogeneous polynomial since the sequence of adjoint ideals of a coherent sequence is again coherent 3.1.30. We have to prove that Q belongs to $\mathfrak{A}_l^*(E) = \left(\bigotimes_{\alpha_l}^{l,s} E\right)'$ and that $\|Q\|_{\mathfrak{A}_l^*(E)} \leq M^k\|P\|_{\mathfrak{A}_k^*(E)}$. Take $s = \sum_j y_j^l \in \bigotimes_{\alpha_l}^{l,s} E$ and $\varepsilon > 0$. Since $\varphi \in \left(\bigotimes_{\alpha_{k-l}}^{k-l,s} E\right)''$, by Goldstine theorem there is some $t = \sum_i x_i^{k-l} \in \bigotimes_{\alpha_{k-l}}^{k-l,s} E$, with $\alpha_{k-l}(t) \leq 1$ such that $|\langle Q, s\rangle| = |\sum_j \varphi(P_{y_j^l})| \leq |\langle \sum_j P_{y_j^l}, t\rangle| + \varepsilon$. Thus,

$$\begin{aligned} |\langle Q, s\rangle| &\leq |\langle \sum_j P_{y_j^l}, \sum_i x_i^{k-l}\rangle| + \varepsilon = \big|\sum_{i,j} \overset{\vee}{P}(x_i^{k-l}, y_j^l)\big| + \varepsilon = \big|\langle P, \sigma\big(\sum_{i,j} x_i^{k-l} \otimes y_j^l\big)\rangle\big| + \varepsilon \\ &\leq \|P\|_{\mathfrak{A}_k^*(E)}\alpha_k(\sigma(s \otimes t)) + \varepsilon \leq M^k\|P\|_{\mathfrak{A}_k^*(E)}\alpha_l(s)\alpha_{k-l}(t) + \varepsilon. \end{aligned}$$

Since this is true for arbitrarily small ε and $\alpha_{k-l}(t) \leq 1$, we conclude that $\|Q\|_{\mathfrak{A}_l^*(E)} \leq M^k\|P\|_{\mathfrak{A}_k^*(E)}$. $\square$

Next corollary to Proposition 4.1.17 follows, for maximal ideals, from the equality $\mathfrak{A}_k^{**} = \mathfrak{A}_k$. For minimal ideals is a consequence of Corollary 3.2.27.

Corollary 4.1.18. *Let $\{\mathfrak{A}_k\}_k$ be sequence of maximal polynomial ideals or minimal polynomial ideals. Then $\{\mathfrak{A}_k\}_k$ is weakly differentiable (multiplicative) if and only if $\{\mathfrak{A}_k^*\}_k$ is multiplicative (weakly differentiable).*

Corollary 4.1.19. *The sequence $\{\mathcal{P}_I^k\}$ of ideals of integral polynomials is weakly differentiable with constant $K = 1$.*

Proof. This is a consequence of the above proposition (ii), since $\mathcal{P}_I^k = (\mathcal{P}^k)^*$ and $\{\mathcal{P}^k\}$ is multiplicative with constant $M = 1$. □

4.2 The convolution product for sequences of minimal ideals. Hypercyclicity

In section 3.2.2 we defined weakly differentiable sequences of polynomial ideals and we showed that if $\mathfrak{A}$ is a weakly differentiable sequence then there is a well defined convolution product on $H_{b\mathfrak{A}}(E)'$. In this section will show that if $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ (via the Borel transform as in section 3.2.3, see Remark 3.2.28) and the sequence $\{\mathfrak{B}_k\}_k$ is multiplicative at E' then $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$ is weakly differentiable. Recall that, if $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$, there is a vector isomorphism between $H_{b\mathfrak{A}}(E)'$ and $Exp_{\mathfrak{B}}(E')$ (Proposition 3.2.35). We will prove that the convolution product on $H_{b\mathfrak{A}}(E)'$ may be characterized as pointwise multiplication on $Exp_{\mathfrak{B}}(E')$, so that the isomorphism becomes an algebra isomorphism. This will allow us to obtain another characterization of convolution operators on $H_{b\mathfrak{A}}(E)$ and a slightly different statement of the Godefroy-Shapiro Theorem.

Lemma 4.2.1. *Let $\{\mathfrak{B}_k(E')\}_k$ be a multiplicative sequence and suppose $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then $\mathfrak{A}$ is weakly differentiable at E, that is, if $P \in \mathfrak{A}_k(E)$ and $\varphi \in \mathfrak{A}_{k-l}(E)'$, $k \geq l$. Then the l-homogeneous polynomial $x \mapsto \varphi(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and*

$$\|x \mapsto \varphi(P_{x^l})\|_{\mathfrak{A}_l(E)} \leq M^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Proof. If P is a finite type polynomial then $x \mapsto \varphi(P_{x^l})$ is also a finite type polynomial and thus belongs to $\mathfrak{A}_l(E)$. We can therefore define a linear operator

$$\begin{array}{rccc} T: & \left(\mathcal{P}_f^k(E), \|\cdot\|_{\mathfrak{A}_k(E)}\right) & \to & \mathfrak{A}_l(E) \\ & P = \sum_{j=1}^N \gamma_j^k & \mapsto & \Big[x \mapsto \varphi(P_{x^l})\Big]. \end{array}$$

If $P = \sum_{j=1}^N \gamma_j^k$ and $\psi \in \mathfrak{A}_l(E)'$ then

$$\psi(T(P)) = \sum_{j=1}^N (B_{k-l}(\varphi) B_l(\psi))(\gamma_j) = B_k^{-1}(B_{k-l}(\varphi) B_l(\psi))(P).$$

Then, for every $\psi \in \mathfrak{A}_l(E)'$,

$$|\psi(T(P))| \leq M^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|\psi\|_{\mathfrak{A}_l(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Therefore, T is continuous and, by Lemma 3.2.29, can be extended to $\mathfrak{A}_k(E)$. By density, it easily follows that $T(P)(x) = \varphi(P_{x^l})$ for every $x \in E$ and every $P \in \mathfrak{A}_k(E)$. □

Therefore, the above lemma together with Corollary 3.2.20, show that we can define a convolution product on $H_{b\mathfrak{A}}(E)'$. By Proposition 3.2.35 this space is (vector space) isomorphic to $Exp_{\mathfrak{B}}(E')$. On the other hand, if $\mathfrak{B}(E') = \{\mathfrak{B}_k(E')\}_k$ is a multiplicative sequence then $Exp_{\mathfrak{B}}(E')$ is an algebra with pointwise multiplication. Indeed, if $f, g \in Exp_{\mathfrak{B}}(E')$, with $A_1 = \sup_k \|d^k f(0)\|_{\mathfrak{B}_k(E')}^{\frac{1}{k}}$ and $A_2 = \sup_k \|d^k g(0)\|_{\mathfrak{B}_k(E')}^{\frac{1}{k}}$, we have $d^k(fg)(0) \in \mathfrak{B}_k(E')$ and

$$\|d^k(fg)(0)\|_{\mathfrak{B}_k(E')}^{\frac{1}{k}} \leq M(A_1 + A_2),$$

where M is the multiplicative constant of the sequence $\mathfrak{B}(E')$, that is fg is of $\mathfrak{B}$-exponential type.

This fact allows us to introduce another product on $H_{b\mathfrak{A}}(E)'$, just copying the pointwise multiplication in $Exp_{\mathfrak{B}}(E')$ via the Borel transform:

If $\{\mathfrak{B}_k(E')\}_k$ be a coherent multiplicative sequence and let $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for all k. For $\varphi, \psi \in H_{b\mathfrak{A}}(E)'$ we define the product $\odot$ in $H_{b\mathfrak{A}}(E)'$, by

$$\varphi \odot \psi = \beta^{-1}\big(\beta(\varphi)\beta(\psi)\big).$$

Next we show that the two products defined on $H_{b\mathfrak{A}}(E)'$ are actually the same.

Proposition 4.2.2. *Let $\{\mathfrak{B}_k(E')\}_k$ be a multiplicative sequence and let $\{\mathfrak{A}_k(E)\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Let $\varphi, \psi \in H_{b\mathfrak{A}}(E)'$. Then $\varphi \odot \psi = \varphi * \psi$. As a consequence, there is an algebra isomorphism between $H_{b\mathfrak{A}}(E)'$ and $Exp_{\mathfrak{B}}(E')$.*

Proof. Since finite type polynomials are dense in $H_{b\mathfrak{A}}(E)$, it is sufficient to verify that, for each $\gamma \in E'$ and $k \geq 0$, $\varphi \odot \psi(\gamma^k) = \varphi * \psi(\gamma^k)$.

For $g \in Exp_{\mathfrak{B}}(E')$, $\beta^{-1}(g)(\gamma^k) = B_k^{-1}\big(d^k g(0)\big)(\gamma^k) = d^k g(0)(\gamma)$. Then,

$$\begin{aligned} \varphi \odot \psi(\gamma^k) &= \beta^{-1}(\beta(\varphi)\beta(\psi))(\gamma^k) = d^k\big(\beta(\varphi)\beta(\psi)\big)(0)(\gamma) \\ &= \sum_{j=0}^{k} \binom{k}{j} d^j\big(\beta(\varphi)\big)(0)(\gamma) d^{k-j}\big(\beta(\psi)\big)(0)(\gamma) \\ &= \sum_{j=0}^{k} \binom{k}{j} \varphi(\gamma^j)\psi(\gamma^{k-j}). \end{aligned}$$

On the other hand, since $(\varphi * \gamma^k)(x) = \varphi(\tau_x \gamma^k) = \sum_{j=0}^{k} \binom{k}{j} \varphi(\gamma^j)\gamma(x)^{k-j}$, we obtain

$$\varphi * \psi(\gamma^k) = \psi\big(\varphi * \gamma^k\big) = \sum_{j=0}^{k} \binom{k}{j} \varphi(\gamma^j)\psi(\gamma^{k-j}) = \varphi \odot \psi(\gamma^k).$$

$\square$

As an immediate consequence, we have:

Corollary 4.2.3. *If $\{\mathfrak{B}_k(E')\}_k$ is a multiplicative sequence and $\{\mathfrak{A}_k(E)\}_k$ is such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k then the convolution product in $H_{b\mathfrak{A}}(E)'$ is commutative.*

Using Proposition 4.2.2 it is easy to show the following variation of Corollary 3.2.19 which characterizes convolution operators on $H_{b\mathfrak{A}}(E)$:

Corollary 4.2.4. *Let $\{\mathfrak{B}_k(E')\}_k$ be a multiplicative sequence and let $\{\mathfrak{A}_k(E')\}_k$ be such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. Then $\psi \mapsto T_{\beta^{-1}(\psi)}$ is an algebra isomorphism from $Exp_{\mathfrak{B}}(E')$ onto the algebra of convolution operators on $H_{b\mathfrak{A}}(E)$.*

If $\mathfrak{B}(E')$ is a multiplicative sequence, then the convolution operators on $H_{b\mathfrak{A}}(E)$ are exactly the operators of the form $f \mapsto \varphi * f$ with $\varphi \in H_{b\mathfrak{A}}(E)'$ (Corollary 3.2.19). Therefore, we have the following version of the Godefroy-Shapiro Theorem for $H_{b\mathfrak{A}}$ (Theorem 3.2.39):

Corollary 4.2.5. *Suppose that E' is separable. Let $\{\mathfrak{B}_k(E')\}_k$ be a multiplicative sequence and let $\{\mathfrak{A}_k(E)\}_k$ be a sequence such that $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$ for every k. For every $\varphi \in H_{b\mathfrak{A}}(E)'$ which is not a scalar multiple of δ_0, the operator*

$$\begin{array}{rccc} T_\varphi : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E) \\ & f & \mapsto & T_\varphi(f) = \varphi * f \end{array}$$

is hypercyclic.

Proof. Just note that T_φ is a scalar multiple of the identity if and only if φ is a scalar multiple of δ_0, and use Theorem 3.2.39. □

Example 4.2.6. We may apply the last result to the spaces in Examples 3.2.40 and 3.2.41 of holomorphic functions of compact and nuclear bounded type, since in both cases the sequences $\{\mathfrak{B}_k(E')\}_k$ are multiplicative. The same is true for the sequence of minimal ideals associated to the tensor norm η_k.

By Proposition 4.1.6, the sequence of Schatten-von Neumann polynomials, defined in Subsection 3.2.5, is multiplicative so we can also apply the previous corollary in this case.

Moreover, let us see that the differentiation operators used in [AB99] are convolution operators, and so the result proved there is included in Corollary 4.2.5. Indeed, if $\Phi(z) = \sum c_n z^n$ is an exponential type function and $a \in E$, we define $h(\gamma) = \sum c_n \gamma(a)^n$. Then, $h \in Exp_{\mathfrak{B}}(E')$ and $\beta^{-1}(h)(f) = \sum c_n d^n f(0)(a)$ for all $f \in H_{b\mathfrak{A}}(E)$. Therefore,

$$\beta^{-1}(h) * f(x) = h(\tau_x f) = \sum c_n (d^n \tau_x f)(0)(a) = \sum c_n d^n f(x)(a).$$

That is, the differentiation operator constructed with Φ is $T_{\beta^{-1}(h)}$ and it is thus a convolution operator.

4.3 The spectrum of algebras of entire functions of bounded type

We proved in Lemma 4.1.4 that if $\mathfrak{A}$ is a multiplicative sequence then $H_{b\mathfrak{A}}(E)$ is an algebra. In this section we study the spectrum of this algebra.

In [AGGM96] an analytic structure in the spectrum of $H_b(U)$ (U an open subset of symmetrically regular Banach space) was given and it was shown that the functions in $H_b(U)$ have analytic extension to the spectrum. For the case of entire functions Dineen proved in [Din99, Section 6.3] that the extensions to the spectrum are actually of bounded type in each connected component[2] of the spectrum.

We will show that it is possible to attach an analogous analytic structure to the spectrum $M_{b\mathfrak{A}}(E)$ of $H_{b\mathfrak{A}}(E)$ for a wide class of Banach spaces E and most of the examples of multiplicative sequences $\mathfrak{A}$ considered so far. For this we study the Aron-Berner extension of functions in $H_{b\mathfrak{A}}(E)$ and also translation and convolution operators on these algebras. In this case the spectrum turns out to be a Riemann domain spread over E'' and, as in [AGGM96] or [Din99], each connected

[2]We will prove in Proposition 5.5.2 that this does not imply that the extension is of bounded type on the whole spectrum.

component of $M_{b\mathfrak{A}}(E)$ is an analytic copy of E''. One may thus wonder if the Gelfand extension of a function f to $M_{b\mathfrak{A}}(E)$ is analytic and, also, if the restriction of this extension to each connected component can be thought as a function in $H_{b\mathfrak{A}}(E'')$. Under the additional condition of weakly differentiability of the sequence (which was already defined in Subsection 3.2.2) we will give a positive answer to both questions.

In the next subsection we will apply some of these concepts to derive a Banach-Stone type theorem for the algebras $H_{b\mathfrak{A}}$.

For $\mathfrak{A}$ a multiplicative sequence, let us consider the spectrum $M_{b\mathfrak{A}}(E)$ of the algebra $H_{b\mathfrak{A}}(E)$ (i.e. the set of continuous nonzero multiplicative functionals on $H_{b\mathfrak{A}}$). Since the inclusion $H_{b\mathfrak{A}}(E) \hookrightarrow H_b(E)$ is continuous, evaluations at points of E'' belong to $M_{b\mathfrak{A}}(E)$. Therefore, δ_z is a continuous homomorphism for each $z \in E''$ and we can see E'' as a subset of $M_{b\mathfrak{A}}(E)$.

Also, given $\varphi \in M_{b\mathfrak{A}}(E)$ we can define an element $\pi(\varphi) \in E''$ by $\pi(\varphi)(\gamma) = \varphi(\gamma)$ for every $\gamma \in E'$. Then the linear mapping

$$\begin{aligned} \pi : M_{b\mathfrak{A}}(E) &\rightarrow E'' \\ \varphi &\mapsto \varphi|_{E'} \end{aligned}$$

is a projection from $M_{b\mathfrak{A}}(E)$ onto $E'' \subset M_{b\mathfrak{A}}(E)$. From the definition of π, for $\varphi \in M_{b\mathfrak{A}}(E)$ and $\gamma \in E'$ we have

$$\varphi(\gamma^N) = \varphi(\gamma)^N = \Big(\pi(\varphi)(\gamma)\Big)^N = \Big(AB(\gamma)(\pi(\varphi))\Big)^N = AB(\gamma^N)(\pi(\varphi)).$$

Thus, for every finite type polynomial P,

$$\varphi(P) = AB(P)(\pi(\varphi)) = \delta_{\pi(\varphi)}(P).$$

As a consequence, we have the following:

Lemma 4.3.1. *Let $\mathfrak{A}$ be a multiplicative sequence and E a Banach space such that, for every k, the finite type k-homogeneous polynomials are dense in $\mathfrak{A}_k(E)$. Then $M_{b\mathfrak{A}}(E) = E''$.*

Example 4.3.2. Since the finite type polynomials are dense in any minimal ideal, if $\mathfrak{A}$ is a multiplicative sequence of minimal ideals, then $M_{b\mathfrak{A}}(E) = E''$ for any Banach space E. In particular, this happens for the nuclear and the approximable polynomials, so $M_{bN}(E) = E''$ and $M_{bA}(E) = E''$.

The Aron-Berner extension plays a crucial role in the analytic structure of $M_b(E)$ given in [AGGM96]. In order to obtain a similar structure for our algebras, we need the polynomial ideals to have a good behavior with these extensions. So let us introduce the following:

Definition 4.3.3. A sequence $\{\mathfrak{A}_k(E)\}_k$ of scalar valued ideals of polynomials is said to be ***AB*-closed** if there exist a constant $\alpha > 0$ such that for each Banach space E, $k \in \mathbb{N}$ and $P \in \mathfrak{A}_k(E)$ we have that $AB(P)$ belongs to $\mathfrak{A}_k(E'')$ and $\|AB(P)\|_{\mathfrak{A}_k(E'')} \leq \alpha^k \|P\|_{\mathfrak{A}_k(E)}$, where AB denotes the Aron-Berner extension.

We enumerate some particular examples of sequences that are known to be AB-closed with constant $\alpha = 1$:

Example 4.3.4. The sequence $\mathfrak{A}$ is AB-closed with constant $\alpha = 1$ in the following cases: continuous polynomials $\mathfrak{A} = \{\mathcal{P}^k\}_k$ (see [DG89]), integral polynomials $\mathfrak{A} = \{\mathcal{P}_I^k\}_k$ (see [CZ99]), extendible polynomials $\mathfrak{A} = \{\mathcal{P}_e^k\}_k$ (see [Car99]), weakly continuous on bounded sets polynomials $\mathfrak{A} = \{\mathcal{P}_w^k\}_k$ (see [Mor84]), nuclear polynomials $\mathfrak{A} = \{\mathcal{P}_N^k\}_k$, approximable polynomials $\mathfrak{A} = \{\mathcal{P}_A^k\}_k$.

In [CGa] it is shown that if $\mathfrak{A}_k$ is a maximal or a minimal ideal, then the Aron-Berner extension is an isometry from $\mathfrak{A}_k(E)$ into $\mathfrak{A}_k(E'')$ (see also [CGc]), extending a well known result of Davie and Gamelin [DG89] and analogous results for some particular polynomial ideals. Therefore, any sequence $\{\mathfrak{A}_k(E)\}_k$ of scalar valued ideals of maximal (or minimal) polynomials is AB-closed with constant $\alpha = 1$. Note that all the above examples but $\mathfrak{A} = \{\mathcal{P}_w^k\}_k$ are maximal or minimal, so they are covered by the mentioned result in [CGa].

Example 4.3.5. It is easy to prove that if the sequences $\{\mathfrak{A}_k^0\}_k$ and $\{\mathfrak{A}_k^1\}_k$ are AB-closed with constants α_0 and α_1 respectively, then the interpolated sequence $\{\mathfrak{A}_k^\theta\}_k$ is AB-closed with constant $\alpha_0^{1-\theta}\alpha_1^\theta$.

Example 4.3.6. *If $\mathfrak{A} = \{\mathfrak{A}_k\}$ is AB-closed, and $\mathfrak{C}$ is a maximal operator ideal then $\{\mathfrak{A}_k \circ \mathfrak{C}\}$ is a AB-closed.*

Indeed, if $P \in \mathfrak{A}_k \circ \mathfrak{C}$ factorizes as $P = QT$, then $AB(P) = AB(Q) \circ T''$ by w^*-continuity. Since $\mathfrak{C}$ is maximal, T'' belongs to $\mathfrak{C}$ and $\|T\|_{\mathfrak{C}} = \|T''\|_{\mathfrak{C}}$. Therefore $AB(P)$ belongs to $\mathfrak{A}_k \circ \mathfrak{C}$ and $\|AB(P)\|_{\mathfrak{A}_k\circ\mathfrak{C}} \leq \|AB(Q)\|_{\mathfrak{A}_k}\|T\|_{\mathfrak{C}}$.

Remark 4.3.7. Note that as a consequence of the above definition, if $P \in \mathfrak{A}_k(E)$, $j < k$ and $z \in E''$, then $AB(P)_{z^{k-j}} \in \mathfrak{A}_j(E'')$ and $\|AB(P)_{z^{k-j}}\|_{\mathfrak{A}_j(E'')} \leq (C\|z\|)^{k-j}\alpha^k\|P\|_{\mathfrak{A}_k(E)}$.

Moreover, since the $\mathfrak{A}_k$'s are ideals, if $Q \in \mathfrak{A}_j(E'')$ then $Q \circ J_E \in \mathfrak{A}_j(E)$ and $\|Q \circ J_E\|_{\mathfrak{A}_j(E)} \leq \|Q\|_{\mathfrak{A}_j(E'')}$. Therefore for each $P \in \mathfrak{A}_k(E)$, $AB(P)_{z^{k-j}} \circ J_E \in \mathfrak{A}_j(E)$ and $\|AB(P)_{z^{k-j}} \circ J_E\|_{\mathfrak{A}_j(E)} \leq (C\|z\|)^{k-j}\alpha^k\|P\|_{\mathfrak{A}_k(E)}$.

Also note that if $f \in H_{b\mathfrak{A}}(E)$ then $AB(f) \in H_{b\mathfrak{A}}(E'')$ and $p_R(AB(f)) \leq p_{\alpha R}(f)$.

We want now to define a topology on $M_{b\mathfrak{A}}(E)$ which makes $(M_{b\mathfrak{A}}(E), \pi)$ into a Riemann domain. First recall from Lemma 3.2.3 that if $\mathfrak{A}$ is a coherent sequence then

$$\begin{array}{rccc}\tau_x : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E) \\ & f & \mapsto & f(x+\cdot)\end{array}$$

is a continuous operator. Therefore we have,

Lemma 4.3.8. *If $\mathfrak{A}$ is a multiplicative sequence and $\varphi \in M_{b\mathfrak{A}}(E)$ then $\varphi \circ \tau_x \in M_{b\mathfrak{A}}(E)$.*

If $\mathfrak{A}$ is AB-closed and coherent and $z \in E''$ we can define

$$\begin{array}{rccc}\tilde{\tau}_z : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E) \\ & f & \mapsto & \tau_z(AB(f)) \circ J_E.\end{array}$$

Remark 4.3.7 ensures that $\tilde{\tau}_z$ is a (well-defined) continuous operator and $p_R(\tilde{\tau}_z f) \leq p_{\alpha(R+C\|z\|)}(f)$.

Corollary 4.3.9. *Let $\mathfrak{A}$ be an AB-closed and multiplicative sequence and let $z \in E''$. Then $\tilde{\tau}_z$ is a continuous operator. Consequently, if $\varphi \in H_{b\mathfrak{A}}(E)'$ then $\varphi \circ \tilde{\tau}_z \in H_{b\mathfrak{A}}(E)'$ and if $\varphi \in M_{b\mathfrak{A}}(E)$ then $\varphi \circ \tilde{\tau}_z \in M_{b\mathfrak{A}}(E)$.*

Note that $\pi(\varphi \circ \tilde{\tau}_z)(\gamma) = \varphi \circ \tilde{\tau}_z(\gamma) = \varphi(AB(\gamma)(z + J_E(\cdot)) = \varphi(1)z(\gamma) + \varphi(\gamma) = z(\gamma) + \varphi(\gamma)$, and thus $\pi(\varphi \circ \tilde{\tau}_z) = z + \pi(\varphi)$.

In [AGGM96, Proposition 2.3], a necessary condition to obtain the analytic structure of the spectrum of $H_b(U)$ is that the space E be symmetrically regular (i.e. the Arens extensions of every symmetric multilinear form are symmetric). In our case, to study the spectrum of $H_{b\mathfrak{A}}(E)$, we need that the Arens extensions of $\overset{\vee}{P}$ be symmetric for every P in $\mathfrak{A}_k(E)$ (and for all k). This happens, of course, if E is symmetrically regular, but also for arbitrary E if $\mathfrak{A}_k$ are good enough. So we define:

Definition 4.3.10. *A sequence $\mathfrak{A}$ is regular at E if, for every k and every P in $\mathfrak{A}_k(E)$, we have that every Arens extension (that is, the extensions by w^*-continuity, in any order) of $\overset{\vee}{P}$ are symmetric. We say that the sequence $\mathfrak{A}$ is regular if it is regular at E for every Banach space E.*

Example 4.3.11. (a) Any sequence of ideals contained in the ideals of approximable polynomials is regular. In particular, any sequence of minimal ideals is regular.

(b) The sequences of integral [CL04, Proposition 2.14], extendible [CL04, Proposition 2.15] and weakly continuous [AHV83] multilinear forms are regular.

(c) If $\{\alpha_k\}_k$ is a sequence of projective symmetric tensor norms and $\mathfrak{A}$ is a sequence of ideals associated to $\{\alpha_k\}_k$, then $\mathfrak{A}$ is regular. Indeed, if we denote $\beta_k = \alpha_k'$, since $\beta_k \leq \eta_k$, every $P \in \left(\bigotimes_{\beta_k}^{k,s} E\right)'$ is extendible and so it satisfies that $AB(\overset{\vee}{P})$ is symmetric. This says that the sequence of maximal ideals associated to $\{\alpha_k\}_k$ is regular and so is $\mathfrak{A}$.

(d) Let $\{\alpha_k\}_k$ be a sequence of symmetric tensor norms and let $\mathfrak{A}$ be the sequence of maximal polynomial ideals associated to $\{\alpha_k\}_k$. If $\mathfrak{A}$ is regular then is regular also any sequence of polynomial ideals associated to $\{/\alpha_k\backslash\}_k$. Indeed, if we denote $\beta_k = \alpha_k'$, for any $P \in \left(\otimes_{\backslash\beta_k/}^{k,s} E\right)'$ we have that $Q = P \circ \otimes^k q \in \left(\otimes_{\beta_k}^{k,s} \ell_1(B_E)\right)' = \mathfrak{A}_k(\ell_1(B_E))$, where q is the metric projection $\ell_1(B_E) \twoheadrightarrow E$. We claim that any Arens extension of $\overset{\vee}{P}$ is symmetric. Indeed, let σ be a permutation of $\{1, \ldots, k\}$ and let $\widetilde{P}$, $\widetilde{Q}$ the Arens extensions of $\overset{\vee}{P}$, $\overset{\vee}{Q}$ defined by w^*-continuity in the order determined by σ. For $j = 1, \ldots, k$, take $z_j \in E''$ and bounded nets $(x_\lambda^j)_\lambda \subset E$ such that $x_\lambda^j \overset{w^*}{\to} z_j$. Then $\widetilde{Q}(z_1, \ldots, z_k) = \lim_{x_\lambda^{\sigma(1)}} \ldots \lim_{x_\lambda^{\sigma(k)}} \overset{\vee}{Q}(x_\lambda^1, \ldots, x_\lambda^k) = \lim_{x_\lambda^{\sigma(1)}} \ldots \lim_{x_\lambda^{\sigma(k)}} \overset{\vee}{P}(q(x_\lambda^1), \ldots, (x_\lambda^k)) = \widetilde{P}(q''(z_1), \ldots, q''(z_k))$. Since q'' is surjective and $\widetilde{Q}$ is symmetric, we conclude that $\widetilde{P}$ is symmetric.

(e) As a consequence of (c) and (d) we obtain that if $\mathfrak{A}$ is a sequence of polynomial ideals associated with any of the natural sequences (except for the case $\alpha_k = \varepsilon_k$) then $\mathfrak{A}$ is regular.

(f) If the sequences $\{\mathfrak{A}_k^0\}_k$ and $\{\mathfrak{A}_k^1\}_k$ are regular then the interpolated sequence $\{\mathfrak{A}_k^\theta\}_k$ is also regular (because each $\mathfrak{A}_k^\theta$ is contained in $\mathfrak{A}_k^0 + \mathfrak{A}_k^1$).

(g) If the sequence $\{\mathfrak{A}_k\}_k$ is regular and $\mathfrak{C}$ is an operator ideal, then $\{\mathfrak{A}_k \circ \mathfrak{C}\}_k$ is regular because, if a polynomial factorizes, $P = QT$, with $Q \in \mathfrak{A}_k$ and $T \in \mathfrak{C}$ then $AB(\overset{\vee}{P}) = AB(\overset{\vee}{Q}) \circ T''$ which is symmetric.

As in [AGGM96] it can be proved that, if $\mathfrak{A}$ is regular at E then every evaluation at a point of E^{iv} is in fact an evaluation at a point of E''.

The next two lemmas can be obtained just as in [Din99, Pages 428-430].

Lemma 4.3.12. *Let $\mathfrak{A} = \{\mathfrak{A}_k(E)\}_k$ be an AB-closed coherent sequence which is regular at a Banach space E. Then $\tilde{\tau}_z \circ \tilde{\tau}_w = \tilde{\tau}_{z+w}$ for every $z, w \in E''$.*

Lemma 4.3.13. *Let $\mathfrak{A} = \{\mathfrak{A}_k(E)\}_k$ be an AB-closed multiplicative sequence which is regular at a Banach space E. For each $\varphi \in M_{b\mathfrak{A}}(E)$ and $\varepsilon > 0$ define $V_{\varphi,\varepsilon} = \{\varphi \circ \tilde{\tau}_z : z \in E'', \|z\| < \varepsilon\}$. Then $\left\{V_{\varphi,\varepsilon} : \varphi \in M_{b\mathfrak{A}}(E), \varepsilon > 0\right\}$ is the basis of a Hausdorff topology in $M_{b\mathfrak{A}}(E)$.*

Proposition 4.3.14. *Let $\mathfrak{A}$ be an AB-closed multiplicative sequence which is regular at a Banach space E. Then $(M_{b\mathfrak{A}}(E),\pi)$ is a Riemann domain over E'' and each connected component of $(M_{b\mathfrak{A}}(E),\pi)$ is homeomorphic to E''.*

Proof. With the topology defined in the above lemma, it is clear that for each $\varphi \in M_{b\mathfrak{A}}(E)$ and $\varepsilon > 0$, $\pi|_{V_{\varphi,\varepsilon}}$ is an homeomorphism onto $B_{E''}(\pi(\varphi),\varepsilon)$. Thus $\pi : M_{b\mathfrak{A}}(E) \to E''$ is a local homeomorphism. Note that given $\varphi \in M_{b\mathfrak{A}}(E)$, by Corollary 4.3.9, $\varphi \circ \tilde{\tau}_z$ is an homomorphism for each $z \in E''$. Moreover, since $\pi(\varphi \circ \tilde{\tau}_z) = \pi(\varphi) + z$ it follows that π is an homeomorphism from $S(\varphi) := \{\varphi \circ \tilde{\tau}_z : z \in E''\}$ to E'' and thus $S(\varphi)$ is the connected component of φ in $M_{b\mathfrak{A}}(E)$. □

Example 4.3.15. $(M_{b\mathfrak{A}}(E),\pi)$ is a Riemann domain over E'' (and each connected component is homeomorphic to E'') in the following cases:

(a) $\mathfrak{A} = \{\mathcal{P}_I^k\}_k$ or $\mathfrak{A} = \{\mathcal{P}_e^k\}_k$ or, more generally, $\mathfrak{A}$ the sequence of maximal polynomial ideals associated to any of the natural sequences $\{\alpha_k\}_k$ (except for $\alpha_k = \epsilon_k$) and E any Banach space.

(b) $\mathfrak{A} = \{\mathcal{P}_w^k\}_k$ and E any Banach space.

(c) $\mathfrak{A}$ any multiplicative sequence of maximal polynomial ideals and E symmetrically regular.

(d) $\mathfrak{A} = \{\mathfrak{A}_k^\theta\}_k$, with $\{\mathfrak{A}_k^0\}_k$ and $\{\mathfrak{A}_k^1\}_k$ any of the sequences of the examples (a) or (b) (or (c) and E symmetrically regular).

(e) $\mathfrak{A} = \{\mathfrak{A}_k \circ \mathfrak{C}\}_k$, where $\{\mathfrak{A}_k\}_k$ an AB-closed multiplicative sequence which is regular at a Banach space E and $\mathfrak{C}$ a maximal operator ideal.

Each function $f \in H_{b\mathfrak{A}}(E)$ can be extended via its Gelfand transform $\tilde{f}$ to the spectrum $M_{b\mathfrak{A}}(E)$, that is $\tilde{f}(\varphi) = \varphi(f)$. Now that we have proved that $M_{b\mathfrak{A}}(E)$ is a Riemann domain, it is natural to ask if $\tilde{f}$ is analytic in $M_{b\mathfrak{A}}(E)$. Moreover, one can wonder if $\tilde{f}$ preserve some of the properties of f in terms of the ideals $\mathfrak{A}$.

Given $\varphi \in H_{b\mathfrak{A}}(E)'$ and $f \in H_{b\mathfrak{A}}(E)$, Corollary 4.3.9 allows us to define a function on E'', defined by $z \mapsto \varphi \circ \tilde{\tau}_z(f)$. We will show in Theorem 4.3.17 that this function belongs to $H_{b\mathfrak{A}}(E'')$. This will allow us to conclude that the restriction of $\tilde{f}$ to each connected component of $M_{b\mathfrak{A}}(E)$ is $\mathfrak{A}$-holomorphic (Theorem 4.3.19 below). To achieve this we will need an additional condition on the sequence $\mathfrak{A}$: weakly differentiability. This condition was already used in the previous chapter (see Defintion 3.2.15) in order to deal with the convolution product on $H_{b\mathfrak{A}}(E)$. We saw there that there are a lot of weakly differentiable sequences of polynomial ideals.

Remark 4.3.16. If E is a Banach space and $\mathfrak{A}$ is a weakly differentiable coherent sequence which is AB-closed (with constant α), it easily follows that the mapping $E'' \ni z \mapsto \varphi(AB(P)_{z^l} \circ J_E)$ belongs to $\mathfrak{A}_l(E'')$ and

$$\left\| z \mapsto \varphi(AB(P)_{z^l} \circ J_E) \right\|_{\mathfrak{A}_l(E'')} \leq \alpha^k K^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

The proof of the following theorem is similar to Theorem 3.2.17.

Theorem 4.3.17. *Let $\mathfrak{A} = \{\mathfrak{A}_k(E)\}_k$ be an AB-closed weakly differentiable coherent sequence. For each $\varphi \in (H_{b\mathfrak{A}}(E))'$, the following operator is well defined and continuous:*

$$\begin{array}{rccc} \tilde{T}_\varphi : & H_{b\mathfrak{A}}(E) & \to & H_{b\mathfrak{A}}(E'') \\ & f & \mapsto & (z \mapsto \varphi \circ \tilde{\tau}_z(f)) \end{array}$$

Proof. Take $f = \sum_{k=0}^{\infty} P_k \in H_{b\mathfrak{A}}(E)$ and $z \in E''$. T hen $\varphi \circ \tilde{\tau}_z(f) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \varphi\big(AB(P_k)_{z^j} \circ J_E\big) = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \binom{k}{j} \varphi\big(AB(P_k)_{z^j} \circ J_E\big)$ since using Remark 4.3.7 and Remark 3.2.16 it is easy to see that this series is absolutely convergent.

Let $Q_l(z) = \sum_{k=l}^{\infty} \binom{k}{l} \varphi\big(AB(P_k)_{z^l} \circ J_E\big)$. Then $\varphi \circ \tilde{\tau}_z(f) = \sum_{l=0}^{\infty} Q_l(z)$. We will show that Q_l belongs to $\mathfrak{A}_l(E'')$ and that $\sum_{l=0}^{\infty} Q_l$ is in $H_{b\mathfrak{A}}(E'')$. To prove this it suffices to show that the series $\sum_{k=l}^{\infty} \binom{k}{l} \Big\| z \mapsto \varphi\big(AB(P_k)_{z^l} \circ J_E\big) \Big\|_{\mathfrak{A}_l(E'')}$ converges and that for every $R > 0$, the series $\sum_{l=0}^{\infty} R^l \Big\| \sum_{k=l}^{\infty} \binom{k}{l} z \mapsto \varphi\big(AB(P_k)_{z^l} \circ J_E\big) \Big\|_{\mathfrak{A}_l(E'')}$ also converges. By Remark 4.3.16 we have

$$\begin{aligned}
\sum_{l=0}^{\infty} R^l \Big\| \sum_{k=l}^{\infty} \binom{k}{l} z \mapsto \varphi\big(AB(P_k)_{z^l} \circ J_E\big) \Big\|_{\mathfrak{A}_l(E'')} &\leq \sum_{l=0}^{\infty} R^l \sum_{k=l}^{\infty} \binom{k}{l} \Big\| z \mapsto \varphi\big(AB(P_k)_{z^l} \circ J_E\big) \Big\|_{\mathfrak{A}_l(E'')} \\
&\leq \sum_{l=0}^{\infty} R^l \sum_{k=l}^{\infty} \binom{k}{l} \alpha^k K^k \|\varphi_{|_{\mathfrak{A}_{k-l}(E)}}\|_{\mathfrak{A}_{k-l}(E)'} \|P_k\|_{\mathfrak{A}_k(E)} \\
&\leq c \sum_{k=0}^{\infty} \alpha^k \|P_k\|_{\mathfrak{A}_k(E)} K^k \sum_{l=0}^{k} \binom{k}{l} (R)^l r^{k-l} \\
&= c p_{_{\alpha K(R+r)}}(f),
\end{aligned}$$

where in the last inequality we used Remark 3.2.16 and reversed the order of summation. Therefore $\tilde{T}_\varphi(f)$ belongs to $H_{b\mathfrak{A}}(E'')$ and $p_{_R}(\tilde{T}_\varphi(f)) \leq c p_{_{\alpha K(R+r)}}(f)$, that is, $\tilde{T}_\varphi \in \mathcal{L}(H_{b\mathfrak{A}}(E), H_{b\mathfrak{A}}(E''))$. $\square$

In Corollary 3.2.20 we defined the convolution product in $H_{b\mathfrak{A}}(E)'$ when $\mathfrak{A}$ is a weakly differentiable coherent sequence. If $\mathfrak{A}$ is also multiplicative, the convolution is a product on the spectrum:

Corollary 4.3.18. *Let $\mathfrak{A}$ be a weakly differentiable multiplicative sequence. For $\varphi, \psi \in M_{b\mathfrak{A}}(E)$ we can define $\varphi * \psi \in M_{b\mathfrak{A}}(E)$ by $\varphi * \psi(f) = \psi(\varphi * f)$, and the application*

$$\begin{array}{rccc}
M_\varphi : & M_{b\mathfrak{A}}(E) & \to & M_{b\mathfrak{A}}(E) \\
& \psi & \mapsto & \psi * \varphi
\end{array}$$

is continuous.

Now we are ready to prove that the extensions to the spectrum are analytic.

Theorem 4.3.19. *Let E be a Banach space and $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ an AB-closed multiplicative sequence, weakly differentiable and regular at E. Then, for every function $f \in H_{b\mathfrak{A}}(E)$, the extension $\tilde{f}$ to $M_{b\mathfrak{A}}(E)$ results an $\mathfrak{A}$-holomorphic function of bounded type when restricted to each connected component of $M_{b\mathfrak{A}}(E)$.*

Proof. We have to show that for every $\varphi \in M_{b\mathfrak{A}}(E)$, $\tilde{f} \circ \big(\pi|_{S(\varphi)}\big)^{-1} \in H_{b\mathfrak{A}}(E'')$. But note that $S(\varphi) = \{\varphi \circ \tilde{\tau}_z : z \in E''\}$ and that $\big(\pi|_{S(\varphi)}\big)^{-1}(z) = \varphi \circ \tilde{\tau}_{z-\pi(\varphi)}$ so $\tilde{f} \circ \big(\pi|_{S(\varphi)}\big)^{-1}(z) = \varphi \circ \tilde{\tau}_{z-\pi(\varphi)}(f)$. That is, $\tilde{f} \circ \big(\pi|_{S(\varphi)}\big)^{-1} = \widetilde{T}_{\varphi \circ \tilde{\tau}_{-\pi(\varphi)}}(f)$ which is in $H_{b\mathfrak{A}}(E'')$ by Theorem 4.3.17. $\square$

We can apply the last result in the following cases:

Example 4.3.20. $(M_{b\mathfrak{A}}(E), \pi)$ is a Riemann domain over E'' and every function in $H_{b\mathfrak{A}}(E)$ extends to an $\mathfrak{A}$-holomorphic function of bounded type on each connected component of $M_{b\mathfrak{A}}(E)$ in the following cases:

(a) $\mathfrak{A} = \{\mathcal{P}^k\}_k$, and E is symmetrically regular (this is [Din99, Proposition 6.30]).

(b) $\mathfrak{A} = \{\mathcal{P}_I^k\}_k$, for every Banach space E.

(c) $\mathfrak{A} = \{\mathcal{P}_e^k\}_k$, for every Banach space E.

(d) $\mathfrak{A} = \{\mathcal{P}_w^k\}_k$, for every Banach space E.

(e) If $\mathfrak{A}$ is a sequence of maximal polynomial ideals associated to any of the natural sequences (except the case $\{\varepsilon_k\}_k$), for every Banach space E.

(f) $\mathfrak{A} = \{\mathfrak{A}_k \circ \mathfrak{C}\}_k$, where $\{\mathfrak{A}_k\}_k$ an AB-closed, weakly differentiable, multiplicative sequence which is regular at a Banach space E and $\mathfrak{C}$ a maximal operator ideal.

Recall that in Subsection 3.2.6 we defined for any coherent sequence $\mathfrak{A}$ the spaces $H_{d\mathfrak{A}}$ and $H_{b\mathfrak{A}}$ of holomorphic functions on Riemann domains.

Corollary 4.3.21. *Let E be a Banach space and $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ an AB-closed multiplicative sequence, weakly differentiable and regular at E. Then, for every function $f \in H_{b\mathfrak{A}}(E)$, the extension $\tilde{f}$ to $M_{b\mathfrak{A}}(E)$ is in $H_{d\mathfrak{A}}(M_{b\mathfrak{A}}(E))$.*

Proof. By the above theorem, $\tilde{f}$ is in $H_{b\mathfrak{A}}$ of each connected component of $M_{b\mathfrak{A}}(E)$. Since any ball is contained in one of these connected components, we conclude that $\tilde{f}$ is of the class $H_{b\mathfrak{A}}$ on every ball in $M_{b\mathfrak{A}}(E)$. This means that $\tilde{f}$ belongs to $H_{d\mathfrak{A}}(M_{b\mathfrak{A}}(E))$. $\square$

On the other hand, for the case of H_b, we will prove in Proposition 5.5.2 that if there exists a homogeneous polynomial which is not weakly continuous on bounded sets, then its extension to the spectrum $M_b(E)$ is not of bounded type. More specifically, if P is an n-homogeneous polynomial whose restriction to a ball is not weakly continuous at 0, we will show that there exist, for each $k \in \mathbb{N}$, $\varphi_k \in M_b(E)$ such that $\varphi_k(x') = 0$ for every $x' \in E'$. Thus, $\pi(\varphi_k) = 0$ for every k and the set $C = \{\varphi_k : k \in \mathbb{N}\}$ is $M_b(E)$-bounded. But, $|\tilde{P}(\varphi_k)| = |\varphi_k(P)| \geq k^n\varepsilon$ and, therefore, $\|\tilde{P}\|_C = \infty$, which implies that $\tilde{P}$ is not of bounded type on $M_b(E)$. Suppose now that $P \in \mathfrak{A}_n(E)$ is not weakly continuous on bounded sets. Then the φ_k's defined in the proof of Proposition 5.5.2 are in $M_{b\mathfrak{A}}(E)$. Indeed, denote by $p_r^{\mathfrak{A}}$ the seminorms which define the topology of $H_{b\mathfrak{A}}(E)$, and p_r the seminorms for $H_b(E)$. Then since the φ_k's are H_b-continuous there exists $r > 0$ such that $\varphi_k(f) \leq p_r(f)$ for every $f \in H_b(E)$. Now if $f \in H_{b\mathfrak{A}}(E)$, then

$$\varphi_k(f) \leq p_r(f) \leq p_r^{\mathfrak{A}}(f).$$

So $\tilde{P}$ is not of bounded type on $M_{b\mathfrak{A}}(E)$. In particular we have:

Proposition 4.3.22. *Let E be a Banach space and $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ an AB-closed multiplicative sequence regular at E. Suppose that there exists a continuous polynomial on $\mathfrak{A}(E)$ which is not weakly continuous on bounded sets. Then there exists a homogeneous polynomial whose extension to the spectrum $M_{b\mathfrak{A}}(E)$ is not in $H_{b\mathfrak{A}}(M_{b\mathfrak{A}}(E))$.*

4.3.1 A Banach-Stone type result

Now we apply some of our results to obtain a Banach-Stone type theorem for algebras associated to multiplicative sequences of polynomial ideals. We follow a procedure as in similar results in [CGM05]. First, we have:

Lemma 4.3.23. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be multiplicative sequences. Suppose that $\phi : H_{b\mathfrak{B}}(E) \to H_{b\mathfrak{B}}(F)$ is a continuous multiplicative operator and define $g : F'' \to E''$ by $g(z) = \pi(\widetilde{\delta}_z \circ \phi)$. Then, g is holomorphic and for every $\gamma \in E'$, $AB(\gamma) \circ g = AB(\phi\gamma)$. In particular, if finite type polynomials are dense on $\mathfrak{A}_k(E)$ (for every k), then $AB(\phi f) = AB(f) \circ g$ for every $f \in H_{b\mathfrak{A}}(E)$.*

Proof. Denote by $\theta_\phi : M_{b\mathfrak{B}}(F) \to M_{b\mathfrak{A}}(E)$ the restriction of the transpose of ϕ. Then g is just the composition $F'' \xrightarrow{\widetilde{\delta}} M_{b\mathfrak{B}}(F) \xrightarrow{\theta_\phi} M_{b\mathfrak{A}}(E) \xrightarrow{\pi} E''$. If we take $z \in F''$ and $\gamma \in E'$, then $g(z)(\gamma) = \widetilde{\delta}_z(\phi\gamma) = AB(\phi\gamma)(z)$. Thus g is weak*-holomorphic on F'' and therefore holomorphic (see for example [Muj86, Example 8D]).

If $\gamma \in E'$ then $AB(\gamma)(g(z)) = g(z)(\gamma) = AB(\phi\gamma)(z)$. Since ϕ multiplicative and continuous, the last assertion follows. □

Although it is hard for a Banach space E to satisfy that finite type polynomials are dense in $H_b(E)$ (c_0 and Tsirelson like spaces do, but no other classical Banach spaces), it is not so uncommon that finite type polynomials be dense in $H_{b\mathfrak{A}}(E)$ for certain sequences $\mathfrak{A}$ and Banach spaces E. Besides those sequences where finite type polynomials are automatically dense (such as approximable or nuclear polynomials), there are combination of ideals and Banach spaces that make finite type polynomials dense (see Example 4.3.25 below).

Theorem 4.3.24. *(a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be multiplicative sequences, $\mathfrak{B}$ also AB-closed. Suppose that finite type polynomials are dense in $\mathfrak{A}_k(E)$ and on $\mathfrak{B}_k(E'')$ for some Banach space E, for all k. If $H_{b\mathfrak{A}}(E)$ and $H_{b\mathfrak{B}}(F)$ are topologically isomorphic algebras, then E' is isomorphic to F'.*

(b) Let $\mathfrak{A}$ be an AB-closed multiplicative sequence such that finite type polynomials are dense in $\mathfrak{A}_k(E'')$. Then $H_{b\mathfrak{A}}(E)$ and $H_{b\mathfrak{A}}(F)$ are topologically isomorphic algebras if and only if E' and F' are isomorphic.

Proof. a) Suppose that $\phi : H_{b\mathfrak{A}}(E) \to H_{b\mathfrak{B}}(F)$ is an isomorphism. Let $g : F'' \to E''$ and $h : E'' \to F''$ be the applications given by Lemma 4.3.23 for ϕ and ϕ^{-1} respectively. Then $h \circ g$ is the composition

$$F'' \overset{\widetilde{\delta}}{\to} M_{b\mathfrak{B}}(F) \overset{\theta_\phi}{\to} M_{b\mathfrak{A}}(E) \overset{\pi}{\to} E'' \overset{\widetilde{\delta}}{\to} M_{b\mathfrak{A}}(E) \overset{\theta_{\phi^{-1}}}{\to} M_{b\mathfrak{B}}(F) \overset{\pi}{\to} F''.$$

Since $M_{b\mathfrak{A}}(E) = \widetilde{\delta}(E'')$, it follows that $h \circ g = id_{F''}$. Thus $dh(g(0)) \circ dg(0) = id_{F''}$ and therefore F'' is isomorphic to a complemented subspace of E'' which implies that every polynomial in $\mathfrak{B}_k(F'')$ is approximable. Since $\mathfrak{B}$ is AB-closed we can conclude that every polynomial in $\mathfrak{B}_k(F)$ is approximable (if $P \in \mathfrak{B}_k(F)$ then $AB(P) \in \mathfrak{B}_k(F'')$, thus $AB(P)$ is approximable and therefore P is approximable). Now, since $M_{b\mathfrak{B}}(F) = \widetilde{\delta}(F'')$, we can prove similarly that $g \circ h = id_{E''}$, that is, $h = g^{-1}$, and differentiating at $g(0)$ we obtain that E'' is isomorphic to F''.

Since every polynomial on $\mathfrak{B}_k(F)$ is approximable we have that $\phi\gamma$ is weakly continuous on bounded sets for every $\gamma \in E'$ and then $AB(\phi\gamma)$ is w^*-continuous on bounded sets. The identity $g(z)(\gamma) = AB(\phi\gamma)(z)$ shown in Lemma 4.3.23 assures then that g is w^*-w^*-continuous on bounded sets. Similarly, g^{-1} is w^*-w^*-continuous on bounded sets. Moreover, applying [ACG95, Lemma 2.1] to $z \mapsto g(z)(\gamma)$, we obtain that de differential of g at any point is w^*-w^*-continuous (and analogously for g^{-1}). Therefore, the isomorphism between E'' and F'' is the transpose of an isomorphism between F' and E'.

b) One implication follows from (a). For the converse, we can follow the reasoning in [LZ00] to obtain the isomorphism between $\mathfrak{A}_k(E)$ and $\mathfrak{A}_k(F)$, for all k. From that it is derived that $H_{b\mathfrak{A}}(E)$ and $H_{b\mathfrak{A}}(F)$ are topologically isomorphic. □

Example 4.3.25. Finite type polynomials are dense in the space of integral polynomial on any Asplund Banach space, since in this case, integral and nuclear polynomials coincide (see [Ale85a, BR01, CD00]). Moreover, finite type polynomials are also dense in the space of extendible polynomials on an Asplund space ([CGc]). Then, as a consequence of Theorem 4.3.24 we have: suppose that E and F are Banach spaces, one of them reflexive, and that $H_{b\mathfrak{A}}(E)$ is isomorphic to $H_{b\mathfrak{B}}(F)$, where $\mathfrak{A}$ and $\mathfrak{B}$ can be any of the sequence of nuclear, integral, approximable or extendible polynomials. Then F is isomorphic to E.

4.4 Hypermultiplicative sequences and algebras of holomorphic functions on open sets

In the previous sections we introduced multiplicative sequences and used this concept to study algebras of entire function of bounded type. In this section we are interested in algebras of holomorphic functions on balls and more general open subsets. We will need to introduce the following more restrictive version of multiplicative sequences.

Definition 4.4.1. *Let $\{\mathfrak{A}_k(E)\}_k$ be a sequence of scalar valued polynomial ideals. We will say that $\{\mathfrak{A}_k(E)\}_k$ is* **hypermultiplicative** *if it is coherent and for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have that $PQ \in \mathfrak{A}_{k+l}(E)$ and*

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq \frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l}\|P\|_{\mathfrak{A}_k(E)}\|Q\|_{\mathfrak{A}_l(E)}.$$

Remark 4.4.2. Stirling's Formula states that $\frac{n^{n+1/2}}{e^n} \leq n! \leq \frac{n^{n+1/2}}{e^{n-1}}$ for every $n \geq 1$, so we have that

$$\frac{(k+l)^{k+l}}{(k+l)!}\frac{k!}{k^k}\frac{l!}{l^l} \leq e^2\Big(\frac{kl}{k+l}\Big)^{1/2}.$$

Thus if $\mathfrak{A}$ is hypermultiplicative, for each $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that for every $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, $\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq c_\varepsilon(1+\varepsilon)^{k+l}\|P\|_{\mathfrak{A}_k(E)}\|Q\|_{\mathfrak{A}_l(E)}$. That is, we could say that $\mathfrak{A}$ hypermultiplicative if it is "almost" multiplicative with constant M for every $M > 1$.

We will show below that most commonly used examples of polynomial ideals form hypermultiplicative sequences. Let us see before that if $\mathfrak{A}$ is hypermultiplicative, then $H_{b\mathfrak{A}}(B_E)$ is an algebra.

In Lemma 4.1.4 we proved that if $\mathfrak{A}$ is a multiplicative sequence, then $H_{b\mathfrak{A}}(E)$ is a B_0-algebra. We will prove that if $\mathfrak{A}$ is hypermultiplicative then $H_{b\mathfrak{A}}(E)$ and $H_{b\mathfrak{A}}(B_E)$ are locally m-convex Fréchet algebras, that is, the topology may be given by submultiplicative seminorms.

Proposition 4.4.3. *Suppose that $\mathfrak{A}$ is hypermultiplicative and E a Banach space. Then,*

(i) for each $x \in E$ and $r > 0$, $H_{b\mathfrak{A}}(B_r(x))$ is a locally m-convex Fréchet algebra.

(ii) $H_{b\mathfrak{A}}(E)$ is a locally m-convex Fréchet algebra.

Proof. (i) We will show this for $r = 1$ and $x = 0$, that is $B_r(x) = B_E$. The general case follows by translation and dilation. We already know that $H_{b\mathfrak{A}}(B_E)$ is a Fréchet space. Let us first show that it is a B_0-algebra.

Let $f = \sum_k P_k$ and $g = \sum_k Q_k$ be functions in $H_{b\mathfrak{A}}(B_E)$. We must show that $\frac{d^n fg(0)}{n!}$ belongs to $\mathfrak{A}_n(E)$ and that $p_s(fg) = \sum_{n=0}^\infty s^n\big\|\frac{d^n fg(0)}{n!}\big\|_{\mathfrak{A}_n(E)} < \infty$ for every $s < 1$. Since $\frac{d^n fg(0)}{n!} =$

$\sum_{k=0}^{n} P_k Q_{n-k}$ and $\mathfrak{A}$ is hypermultiplicative, $\frac{d^n fg(0)}{n!}$ belongs to $\mathfrak{A}_n(E)$. On the other hand, using Stirling's Formula,

$$\begin{aligned}
\sum_{k=0}^{\infty} s^n \Big\|\frac{d^n fg(0)}{n!}\Big\|_{\mathfrak{A}_n(E)} &\leq e^2 \sum_{n=0}^{\infty} s^n \sum_{k=0}^{n} \Big(\frac{k(n-k)}{n}\Big)^{1/2} \|P_k\|_{\mathfrak{A}_k(E)} \|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)} \\
&= e^2 \sum_{k=0}^{\infty} \sqrt{k} s^k \|P_k\|_{\mathfrak{A}_k(E)} \sum_{n=k}^{\infty} s^{n-k} \Big(\frac{n-k}{n}\Big)^{1/2} \|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)} \\
&\leq e^2 p_s(g) \sum_{k=0}^{\infty} \sqrt{k} s^k \|P_k\|_{\mathfrak{A}_k(E)}.
\end{aligned}$$

Therefore, for each $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, s) > 1$ such that

$$p_s(fg) = \sum s^n \Big\|\frac{d^n fg(0)}{n!}\Big\|_{\mathfrak{A}_n(E)} \leq c p_s(g) p_{s+\varepsilon}(f).$$

Therefore, if $f, g \in H_{b\mathfrak{A}}(B_E)$, $s_n = 1 - \frac{1}{2^n}$, for some $n > 1$, and $\varepsilon = \frac{1}{2^{n+1}}$ then if we denote $c_n = c(\frac{1}{2^{n+1}}, 1 - \frac{1}{2^n})$,

$$p_{s_n}(fg) \leq c_n p_s(g) p_{s_{n+1}}(f) \leq c_n p_{s_{n+1}}(f) p_{s_{n+1}}(g). \tag{4.1}$$

This shows that $H_{b\mathfrak{A}}(B_E)$ is a B_0-algebra. Note also that using (4.1) it follows that

$$p_{s_n}(f^k) \leq c_n^{k-1} p_{s_{n+1}}(f)^{k-1} p_{s_n}(f) \leq c_n^k p_{s_{n+1}}(f)^k.$$

Take now an entire function $h \in H(\mathbb{C})$, $h(z) = \sum a_n z^n$. Then for $f \in H_{b\mathfrak{A}}(B_E)$,

$$p_{s_n}\Big(\sum_{k=N}^{M} a_k f^k\Big) \leq \sum_{k=N}^{M} a_k p_{s_n}(f^k) \leq \sum_{k=N}^{M} a_k \big(c_n p_{s_{n+1}}(f)\big)^k,$$

which goes to 0 as N, M increase because h is an entire function. This means that entire functions operate in $H_{b\mathfrak{A}}(B_E)$. Therefore [MRZ62, Theorem 1] implies that $H_{b\mathfrak{A}}(B_E)$ is locally m-convex.

(ii) follows similarly. □

Let $U \subset E$ be any open subset and $\mathfrak{A}$ coherent sequence. In Section 3.2.6 we defined the spaces $H_{d\mathfrak{A}}(U)$ and $H_{b\mathfrak{A}}(U)$ of $\mathfrak{A}$-holomorphic functions on U. Since $H_{d\mathfrak{A}}(U)$ is the space of holomorphic functions which are of the class $H_{b\mathfrak{A}}$ on each ball contained in U, it is immediate from the previous proposition the following:

Corollary 4.4.4. *Suppose that $\mathfrak{A}$ is hypermultiplicative and let $U \subset E$ be any open subset. Then $H_{d\mathfrak{A}}(U)$ is a locally m-convex algebra.*

The same is true for $H_{b\mathfrak{A}}(U)$:

Corollary 4.4.5. *Suppose that $\mathfrak{A}$ is hypermultiplicative and let $U \subset E$ be an open subset. Then $H_{b\mathfrak{A}}(U)$ is a locally m-convex Fréchet algebra.*

Proof. We know from Proposition 3.2.60 that $H_{b\mathfrak{A}}(U)$ is a Fréchet space. Let us see that it is an algebra. Let $A \subset\subset U$, A open and let $\varepsilon > 0$ such that $A + B_\varepsilon(0) \subset\subset U$. By (the proof of) Proposition 4.4.3, if $B_s(x) \subset U$,

$$p_s^x(fg) \leq c_\varepsilon p_s^x(g) p_{s+\varepsilon}^x(f).$$

Note also that $B_s(x) \subset A$ if and only if $B_{s+\varepsilon}(x) \subset A + B_\varepsilon(0)$, therefore

$$p_A(fg) = \sup_{B_s(x)\subset A} p_s^x(gf) \leq c_\varepsilon \Big(\sup_{B_s(x)\subset A} p_s^x(g) \Big) \cdot \Big(\sup_{B_{s+\varepsilon}(x)\subset A+B_\varepsilon(0)} p_{s+\varepsilon}^x(f) \Big) = p_A(g) p_{A+B_\varepsilon(0)}(f) < \infty$$

This shows that $H_{b\mathfrak{A}}(U)$ is a B_0-algebra. We may prove as in Proposition 4.4.3 that entire functions operate in $H_{b\mathfrak{A}}(U)$ and thus by [MRZ62, Theorem 1], $H_{b\mathfrak{A}}(U)$ is locally m-convex. □

We will finish this chapter with examples of hypermultiplicative sequences.

Example 4.4.6. It is immediate to verify that the following sequences are all hypermultiplicative, since they are multiplicative with constant $M = 1$:

i) $\{\mathcal{P}^k\}$, of all homogeneous polynomials,
ii) $\{\mathcal{P}_w^k\}$, of weakly continuous on bounded sets polynomials,
iii) $\{\mathcal{P}_A^k\}$, of approximable polynomials,
iv) $\{\mathcal{P}_e^k\}$, of extendible polynomials,
v) $\{\mathcal{M}_r^k\}$, of multiple r-summing polynomials.

Example 4.4.7. *The sequence $\{\mathcal{P}_I^k\}$ of integral polynomials is hypermultiplicative.*

This was already proved in Example 4.1.5 (c).

Example 4.4.8. *The sequence $\{\mathcal{P}_N^k\}$ of nuclear polynomials is hypermultiplicative.*

Indeed, it is clear that the product of nuclear polynomials is nuclear. Moreover, since nuclear polynomials are the minimal ideal associated to integral polynomials (see for example [Flo01, 3.4]), the previous example implies, by Proposition 4.1.12 that if $P \in \mathcal{P}_N(^kE)$, $Q \in \mathcal{P}_N(^lE)$ then $\|PQ\|_N \leq \frac{(k+l)^{k+l}}{(k+l)!} \frac{k!}{k^k} \frac{l!}{l^l} \|P\|_N \|Q\|_N$. As a consequence of Proposition 4.4.3, the space of nuclearly entire functions of bounded type is a locally m-convex Fréchet algebra.

This can be deduced also from the following set of examples. Recall from Subsection 3.1.5 the sequences of polynomial ideals associated to the natural tensor norms defined in [CGb].

Proposition 4.4.9. *Let $\{\alpha_k\}_k$ be any of the sequences of natural symmetric tensor norms. Then the sequences $\{\mathfrak{A}_k^{max}\}_k$ and $\{\mathfrak{A}_k^{min}\}_k$ of maximal and minimal associated ideals are hypermultiplicative.*

Proof. This follows form the inequalities

$$\pi_{k+l}^s(\sigma(s \otimes t)) \leq \frac{(k+l)^{k+l}}{(k+l)!} \frac{k!}{k^k} \frac{l!}{l^l} \pi_k^s(s) \pi_l^s(t) \text{ and } \varepsilon_{k+l}^s(\sigma(s \otimes t)) \leq \varepsilon_k^s(s) \varepsilon_l^s(t)$$

for every $s \in \bigotimes^{k,s} E'$, $t \in \bigotimes^{l,s} E'$ and Lemma 4.1.15 together with Proposition 4.1.12. □

With the same proof as Proposition 4.1.6 we have that interpolation of hypermultiplicative sequences is again hypermultiplicative.

Chapter 5

Envelopes of holomorphy and extension of functions of bounded type

In this chapter we characterize the envelope of holomorphy for the algebra of bounded type holomorphic functions on Riemann domains over a Banach space in terms of the spectrum of the algebra. We prove that evaluations at points of the envelope are always continuous but we show an example of a balanced open subset of c_0 where the extensions to the envelope are not necessarily of bounded type, answering a question posed by Hirschowitz in 1972. We show that for bounded balanced sets the extensions are of bounded type. We also consider extensions to the bidual, and show some properties of the spectrum in the case of the unit ball of ℓ_p.

The contents of this chapter belong mostly to [CM].

5.1 Envelopes of holomorphy

In this section we obtain a characterization of the envelope of holomorphy for the functions of bounded type on a Riemann domain.

Let us recall the definition of extension morphism and envelope of holomorphy for a family of holomorphic functions (see, for example, [Muj86, Chapter XIII]). Let (X, p) and (Y, q) be Riemann domains spread over a Banach space E. A morphism is a continuous mapping $\tau : X \to Y$ such that $q \circ \tau = p$.

$$\begin{tikzcd} X \arrow[rr, "\tau"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & E & \end{tikzcd}$$

Let $\mathcal{F}$ be a subset of $H(X)$, then a morphism $\tau : X \to Y$ is said to be an $\mathcal{F} - extension$ of X if for each $f \in \mathcal{F}$ there is a unique $\tilde{f} \in H(Y)$ such that $\tilde{f} \circ \tau = f$.

A morphism $\tau : X \to Y$ is said to be an $\mathcal{F} - envelope\ of\ holomorphy$ of X if τ is an $\mathcal{F}$-extension of X and if for each $\mathcal{F}$-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\mu \circ \nu = \tau$. The $\mathcal{F}$-envelope of holomorphy of X can be thought as the largest Riemann domain Y to which every $f \in \mathcal{F}$ has a unique holomorphic extension.

$$\begin{tikzcd}
 & & Y \arrow[ddl, bend left, "q"] \\
X \arrow[r, "\nu"] \arrow[rru, bend left, "\tau"] \arrow[dr, "p"'] & Z \arrow[ur, dashed, "\mu"] \arrow[d] & \\
 & E &
\end{tikzcd}$$

Regarding holomorphic functions of bounded type, the H_b-envelope of holomorphy was constructed, for example, by Hirschowitz in [Hir72]. For general families of functions $\mathcal{F}$, the existence of the $\mathcal{F}$-envelope of holomorphy can be seen in [Muj86, Chapter XIII]. For related characterizations of the envelope of holomorphy in terms of the spectrum for wide classes of holomorphic functions see [Sch72] and [Coe74, Chapter VI].

The functions of bounded type form a class of functions that can be defined on any Riemann domain, and that has a topology different from the space of all holomorphic functions. These two facts may arise some concerns about the proper definition of envelope of holomorphy. For example, it may be more natural to consider the largest Riemann domain to which every $f \in H_b(X)$ has a unique holomorphic extension which is of bounded type. Or the largest Riemann domain Y to which every $f \in H_b(X)$ has a unique holomorphic extension, so that evaluating the extensions on elements of Y are continuous homomorphisms on $H_b(X)$. Note that in several complex variables, the envelope of holomorphy may be identified with the spectrum. If we expect to obtain something similar, evaluations in elements of the envelope must be continuous. This motivates the following definition:

Definition 5.1.1. *Let (X,p), (Y,q) be Riemann domains spread over a Banach space E and let $\mathcal{F} \subset H(X)$ be a topological algebra. A morphism $\tau : X \to Y$ is said to be a strong $\mathcal{F}$-extension of X if for each $f \in \mathcal{F}$ there is a unique $\tilde{f} \in H(Y)$ such that $\tilde{f} \circ \tau = f$ and for each $y \in Y$, the mapping $f \in \mathcal{F} \rightsquigarrow \tilde{f}(y)$ belongs to the spectrum of $\mathcal{F}$.*

The morphism $\tau : X \to Y$ is said to be a strong $\mathcal{F}$-envelope of holomorphy of X if τ is a strong $\mathcal{F}$-extension of X and if for each strong $\mathcal{F}$-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\nu \circ \mu = \tau$.

When there is no confusion, we will say that the (strong) $\mathcal{F}$-envelope of X is the Riemann domain (Y,q).

Now, for the first concern on the extensions being of bounded type (which is probably more natural), we set:

Definition 5.1.2. Let (X,p), (Y,q) be Riemann domains spread over a Banach space E and let $\mathcal{F} \subset H(X)$, $\mathcal{G} \subset H(Y)$. A morphism $\tau : X \to Y$ is said to be an *$\mathcal{F}$-$\mathcal{G}$-extension* of X if for each $f \in \mathcal{F}$ there is a unique $\tilde{f} \in \mathcal{G}$ such that $\tilde{f} \circ \tau = f$.

For the particular case of $\mathcal{F}$ and $\mathcal{G}$ being the spaces of holomorphic functions of bounded type on X and Y, we define:

Definition 5.1.3. Let (X,p) be a Riemann domain spread over a Banach space E. A morphism $\tau : X \to Y$ is said to be an *H_b-H_b-envelope of holomorphy* of X if τ is an H_b-H_b-extension of X and if for each H_b-H_b-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\nu \circ \mu = \tau$.

Finally, we say that a Riemann domain (X, p) is a H_b-domain of holomorphy if it coincides with its H_b-envelope of holomorphy.

It is easy to see that the envelope of holomorphy is, whenever it exists, unique up to an isomorphism. Also, the last definition can be generalized to pairs of classes of functions that are defined in any Riemann domain. It is not clear that any of these variants of the H_b-envelope of holomorphy should necessarily exist. Note that the classical H_b-envelope is known to exist just because it can be constructed [Hir72].

The concept of H_b-extension morphism introduced by Dineen and Venkova in [DV04] is different from ours. The main difference is that in our case, the envelope of a Riemann domain over E is also modeled on E, while theirs is modeled on E'' (just as the spectrum [AGGM96]). The reasons of our choice is that we want to preserve the uniqueness of extensions, as in the finite dimensional setting, and this cannot be achieved if we allow domains on E''. Also, if we want to define domains of holomorphy as those domains that coincide with their envelopes, we need both of them to be modeled over the same space. However, since extensions to the bidual are crucial in the theory of analytic functions of bounded type, we will devote a section to this kind of extensions.

Now we characterize the strong H_b-envelope for a Banach space E, and show that in this case the strong H_b-envelope and the H_b-envelope of X are actually the same. In the next section, we will show that even for balanced open subsets of E, the H_b-H_b-envelope of holomorphy may fail to exist. It does exists if the balanced open subset is also bounded.

As usual, the spectrum of the algebra under consideration plays a crucial role in the study of the envelope of holomorphy. A complex Banach space E is said to be (symmetrically) regular if every continuous (symmetric) linear mapping $T : E \to E'$ is weakly compact. Recall that T is symmetric if $Tx_1(x_2) = Tx_2(x_1)$ for all $x_1, x_2 \in E$. The first steps towards the description of the spectrum $M_b(E)$ of $H_b(E)$ for a symmetrically regular Banach space E were taken by Aron, Cole and Gamelin in their influential article [ACG91]. In [AGGM96, Corollary 2.2] Aron, Galindo, García and Maestre gave $M_b(U)$ a structure of Riemann analytic manifold modeled on E'', for U an open subset of E. For the case $U = E$, $M_b(E)$ can be viewed as the disjoint union of analytic copies of E'', these copies being the connected components of $M_b(E)$. In [Din99, Section 6.3], there is an elegant exposition of all these results. The study of the spectrum of the algebra of the space of holomorphic functions of bounded type was continued in [CGM05]. The analytic structure of $M_b(X)$ for X a Riemann domain over a symmetrically regular Banach space E was presented in [DV04]. The resulting structure for Riemann domains is rather analogous to that of open subsets of E. See Preliminaries 1.3.1 for a short account of the analytic structure of $M_b(X)$.

Note that if E is symmetrically regular, the spectrum $M_b(X)$ is modeled on E'' and for the envelope we require a Riemann domain over E. Next lemma shows that this issue can be fixed for an arbitrary Banach space.

Lemma 5.1.4. *Let (X, p) be a Riemann domain spread over a Banach space E. Then $(\pi^{-1}(E), \pi) \subset (M_b(X), \pi)$ is a Riemann domain spread over E.*

Proof. Let $\varphi \in \pi^{-1}(E)$. If E is symmetrically regular, then by the analytic structure of the spectrum ([AGGM96] or [DV04]), there exist $\delta > 0$ such that if $z \in E''$ and $\|z\| < \delta$, then $\varphi^z \in M_b(X)$ and π is an homeomorphism from $\{\varphi^z : \|z\|_{E''} < \delta\} \subset M_b(X)$ to $B_{E''}(\pi(\varphi), \delta)$. Moreover, $\pi(\varphi^z) = \pi(\varphi) + z$. Thus if $x \in E$ then $\varphi^x \in \pi^{-1}(E)$ and π is a local homeomorphism between $\pi^{-1}(E)$ and E.

In [AGGM96], [DV04] symmetric regularity is used to ensure that the Aron-Berner extension of every symmetric multilinear form is symmetric. But since we restrict to $(\pi^{-1}(E), \pi)$, we can define φ^x as

$$\varphi^x(f) = \sum_{n=0}^{\infty} \varphi\Big(\frac{d^n f(\cdot)}{n!}(x)\Big),$$

and thus we will not make use of the Aron-Berner extension at any moment. Therefore repeating the proofs of [AGGM96] or [DV04] we obtain our result for an arbitrary Banach space E. $\square$

Now we are ready to give the characterization of the strong H_b-envelope of holomorphy, which is very similar to that of several complex variables, especially if E is reflexive:

Theorem 5.1.5. *Let (X, p) be a connected Riemann domain spread over a Banach space E and let Y be the connected component of $\pi^{-1}(E) \subset M_b(X)$ which intersects $\delta(X)$. Then $\delta : (X, p) \to (Y, \pi)$, $\delta(x) = \delta_x$ is the strong H_b-envelope of X.*

Proof. The fact that $\delta : (X, p) \to (Y, \pi)$ is an H_b-extension can be proved as in [DV04, Proposition 2.3].

Let $\tau : (X, p) \to (Z, q)$ be a strong H_b-extension morphism. We must show that there is a morphism $\nu : Z \to Y$ such that $\nu \circ \tau = \delta$.

For $f \in H_{b\mathfrak{A}}(X)$, we denote $\tilde{f}$ its extension to Z. Since for every $z \in Z$, the application $f \in\rightsquigarrow \tilde{f}(z)$ is in $M_b(X)$, there is a well defined mapping

$$\begin{aligned} \nu : \quad & Z \to M_b(X) \\ & \nu(z)(f) = \tilde{f}(z), \end{aligned}$$

for $f \in H_b(X)$, $z \in Z$.

Moreover, $\nu(Z) \subset \pi^{-1}(E)$. Indeed, if $z \in Z$ and $\gamma \in E'$ then

$$\pi(\nu(z))(\gamma) = \nu(z)(\gamma \circ p) = (\gamma \circ p)^{\sim}(z) = \gamma(q(z)). \tag{5.1}$$

Thus $\pi(\nu(z)) = q(z)$ which belongs to E.

Note that (5.1) also proves that $\pi \circ \nu = q$. Therefore, in order to prove that $\nu : Z \to \pi^{-1}(E)$ is a morphism it remains to show that ν is continuous.

For each $z_0 \in Z$ let V_{z_0} be an open neighborhood such that $q_{|V_{z_0}} : V_{z_0} \to q(V_{z_0})$ is an homeomorphism. If we prove that

$$\nu\big((q_{|V_{z_0}})^{-1}(q(z_0) + x)\big) = \nu(z_0)^x, \tag{5.2}$$

for every $x \in E$ with sufficiently small norm, we will have showed that ν is continuous. In fact, (5.2) implies that ν is a local homeomorphism.

Let $f \in H_b(X)$ then

$$\begin{aligned} \nu\big((q_{|V_{z_0}})^{-1}(q(z_0) + x)\big)(f) &= \tilde{f}\big((q_{|V_{z_0}})^{-1}(q(z_0) + x)\big) = \sum_{n\geq 0} \frac{d^n[\tilde{f} \circ (q_{|V_{z_0}})^{-1}]}{n!}\big(q(z_0)\big)(x) \\ &= \sum_{n\geq 0} \frac{d^n \tilde{f}}{n!}(z_0)(x), \end{aligned}$$

and

$$\nu(z_0)^x(f) = \sum_{n\geq 0} \nu(z_0)\Big(\frac{d^n f}{n!}(\cdot)(x)\Big) = \sum_{n\geq 0} \Big(\frac{d^n f}{n!}(\cdot)(x)\Big)^{\sim}(z_0).$$

Therefore it suffices to prove that

$$\left(\frac{d^n f}{n!}(\cdot)(x)\right)^{\sim}(z_0) = \frac{d^n \tilde{f}}{n!}(z_0)(x), \tag{5.3}$$

for every $n \geq 0$ and every $f \in H_b(X)$.

Let $x \in E$. By [AGGM96, p.550] and [Muj86, Corollary 7.18]

$$g(\cdot) = \frac{d^n f}{n!}(\cdot)(x) \in H_b(X),$$

and

$$h(\cdot) = \frac{d^n \tilde{f}}{n!}(\cdot)(x) \in H(Z).$$

Moreover, h is an extension of g to Z. Indeed if $y \in X$ and (V_y, p) is a chart of y such that $(V_{\tau(y)}, q) = (\tau(V_y), q)$ is a chart of $\tau(y)$, then

$$\begin{aligned} h(\tau(y)) &= \frac{d^n[\tilde{f} \circ (q|_{V_{\tau(y)}})^{-1}]}{n!}\big(q(\tau(y))\big)(x) \overset{(*)}{=} \frac{d^n[f \circ (p|_{V_y})^{-1}]}{n!}\big(p(y)\big)(x) \\ &= \frac{d^n f}{n!}(y)(x) = g(y), \end{aligned}$$

where $(*)$ is true because $\tilde{f} \circ (q|_{V_{\tau(y)}})^{-1} = f \circ (p|_{V_y})^{-1}$ since τ is an H_b-extension.

Since $\left(\frac{d^n f}{n!}(\cdot)(x)\right)^{\sim}$ is also an extension of g to Z, we must have that $h = \left(\frac{d^n f}{n!}(\cdot)(x)\right)^{\sim}$. Therefore we have established (5.3) for every $n \geq 0$.

To conclude the proof just note that Z is connected since X is, and thus $\nu : Z \to Y$ is a morphism. □

We will denote by $(\mathcal{E}_b(X), \pi)$ the strong H_b-envelope of holomorphy of X.

When E is reflexive, the envelope of H_b-envelope resembles the envelope of holomorphy for Riemann domains in several complex variables:

Corollary 5.1.6. *Let (X, p) be a connected Riemann domain spread over a reflexive Banach space E. Then the strong H_b-envelope of X is the morphism $\delta : X \to Y$, $\delta(x) = \delta_x$, where Y is the connected component of $M_b(X)$ which contains $\delta(X)$.*

Now we show that our definition of strong H_b-envelope, coincides with the classical definition of H_b-envelope:

Theorem 5.1.7. *Let (X, p) be a connected Riemann domain spread over a Banach space E. Then the H_b-envelope and the strong H_b-envelope of X coincide.*

Proof. Denote by $(\mathcal{E}(X), q)$ the H_b-envelope of X. Then there are a strong H_b-extension $\tau : X \to \mathcal{E}_b(X)$, an H_b-extension $\sigma : X \to \mathcal{E}(X)$ and a morphism $\nu : \mathcal{E}_b(X) \to \mathcal{E}(X)$ such that $\sigma = \nu \circ \tau$.

Let us see that $\nu(\mathcal{E}_b(X))$ is closed in $\mathcal{E}(X)$. Suppose that $y \in \overline{\nu(\mathcal{E}_b(X))} \setminus \nu(\mathcal{E}_b(X))$. Let $W_n = \{\varphi \in \mathcal{E}_b(X) : \varphi \prec X_n\}$, where $\varphi \prec X_n$ means that $|\varphi(f)| \leq \|f\|_{X_n}$ for every $f \in H_{b\mathfrak{A}}(X)$. Then by [AGGM96] (see also [DV04, Proposition 1.5]), $d_X(W_n) \geq \frac{1}{n}$. Therefore we can get a subsequence of integers $(n_k)_k$ and a sequence $(y_k)_k \subset \mathcal{E}_b(X)$ such that $y_k \in W_{n_{k+1}} \setminus W_{n_k}$ and $y_k \to y$. Thus there are functions $f_k \in H_b(X)$ such that $\|f_k\|_{X_{n_k}} < \frac{1}{2^k}$ and $|\tilde{f}_k(y_k)| > k + \sum_{j=1}^{k-1} |\tilde{f}_j(y_k)|$.

Then the series $\sum_{j=1}^{\infty} f_j$ converges to $f \in H_b(X)$ and moreover $\left|\left(\sum_{j=1}^{\infty} f_j\right)^{\sim}(y_k)\right| = \left|\sum_{j=1}^{\infty} \tilde{f}_j(y_k)\right|$ because y_k belongs to $\mathcal{E}_b(X)$ and thus δ_{y_k} is a continuous homomorphism. Therefore

$$|\tilde{f}(y_k)| = \left|\left(\sum_{j=1}^{\infty} f_j\right)^{\sim}(y_k)\right| = \left|\sum_{j=1}^{\infty} \tilde{f}_j(y_k)\right| \geq |\tilde{f}_k(y_k)| - \left|\sum_{j=1}^{k-1} \tilde{f}_j(y_k)\right| - \left|\sum_{j=k+1}^{\infty} \tilde{f}_j(y_k)\right| > k-1,$$

so we have that $|\tilde{f}(y_k)| \to \infty$ and then f cannot be extended to y. This is a contradiction since y belongs to the H_b-envelope of X, $\mathcal{E}(X)$. Thus $\nu(\mathcal{E}_b(X))$ is closed in $\mathcal{E}(X)$.

On the other hand $\nu(\mathcal{E}_b(X))$ is open in $\mathcal{E}(X)$ because ν is a morphism. Therefore $\nu(\mathcal{E}_b(X)) = \mathcal{E}(X)$. □

Corollary 5.1.8. *Let (X,p), (Y,q) be connected Riemann domains spread over a Banach space E and suppose that the morphism $\nu : X \to Y$ is an H_b-extension. Then τ is an strong H_b-extension.*

Proof. Let $\tau : X \to \mathcal{E}(X)$ be the morphism into the envelope of X. By Theorem 5.1.7, $\mathcal{E}(X) = \mathcal{E}_b(X)$, and thus the evaluation at each point of $\mathcal{E}(X)$ is an $H_b(X)$-continuous homomorphism.

On the other hand, there exist a morphism $\mu : Y \to \mathcal{E}(X)$ such that $\mu \circ \nu = \tau$. Thus the evaluation at a point $y \in Y$ coincides with the evaluation at $\mu(y)$ and therefore it is $H_b(X)$-continuous. □

Theorem 5.1.7 says that the envelope is contained in the spectrum. In other words, evaluations on elements of the envelope are always continuous. Of course, the coincidence of the strong and the classical H_b-envelopes also allows us to give a characterization of the latter:

Corollary 5.1.9. *Let (X,p) be a connected Riemann domain spread over a Banach space E and let Y be the connected component of $\pi^{-1}(E)$ which intersects $\delta(X)$. Then $\delta : (X,p) \to (Y,\pi)$, $\delta(x) = \delta_x$ is the H_b-envelope of X.*

The following result is widely known and follows from a straightforward connectedness argument.

Lemma 5.1.10. *Let (X,p), (Y,q) be connected Riemann domains spread over a Banach space E and let $u,v : X \to Y$ be morphisms. Suppose that X is connected, then either $u(x) = v(x)$ for every $x \in X$ or $u(x) \neq v(x)$ for every $x \in X$.*

Proof. Let $x \in X$, then for $\varepsilon > 0$ small, there exist a neighborhood U of x such that $p|_U : U \to B(p(x),\varepsilon)$ is an homeomorphism and a neighborhood $\tilde{U}$ of $u(x)$ such that $q|_{\tilde{U}} : \tilde{U} \to B(q(u(x)),\varepsilon) = B(p(x),\varepsilon)$ is an homeomorphism. Then, since $u|_U = \left(q|_{\tilde{U}}\right)^{-1} \circ p|_U$, it follows that u (and in the same way any morphism of Riemann domains) is a local homeomorphism.

Let $A = \{x \in X : u(x) = v(x)\}$. Take $x \in A$, and let V be a small enough neighborhood of x such that $u : V \to u(V)$, $q : u(V) \to q(u(V))$, $v : V \to v(V)$ and $q : v(V) \to q(v(V))$ are homeomorphisms.

Let $x' \in V$. Since u,v are morphisms, we have that $q(v(x')) = p(x') = q(u(x'))$ and therefore $u(x') = v(x')$ and thus A is open.

Since A is also closed by continuity of u and v we conclude that $A = X$. □

Theorem 5.1.11. *Let (X,p) be a connected Riemann domain spread over a Banach space E. If the H_b-H_b-envelope of X exists, then it coincides with the H_b-envelope $\mathcal{E}_b(X)$ of X.*

Proof. Let (Y,τ) be the H_b-H_b-envelope of X. We put $V_n = \delta(X) \cup W_n^\circ$, where W_n was defined in the proof of Theorem 5.1.7. Then the extension of every function in $H_b(X)$ to V_n is of bounded type. Thus the inclusion $i_n : X \hookrightarrow V_n$ is an H_b-H_b-extension and thus there exist morphisms $\nu_n : V_n \to Y$ such that $\tau = \nu_n \circ i_n$. If $m > n$ then $\nu_m|_{V_n} : V_n \to Y$ is a morphism and since $\nu_n|_{\delta(X)} = \nu_m|_{\delta(X)} = \tau$ we have by Lemma 5.1.10 that $\nu_m|_{V_n} = \nu_n$.

Therefore the application $\nu : \mathcal{E}_b(X) \to Y$, $\nu(x) = \nu_n(x)$ if $x \in V_n$, is well defined and is a morphism since $\nu|_{V_n}$ is a morphism for every $n \in \mathbb{N}$.

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[rr, bend right, dashed, "\tau"'] & \mathcal{E}_b(X) \arrow[r, "\nu"] & Y \end{tikzcd}$$

On the other hand, it is clear that we have an H_b-extension morphism from X to Y and this gives a morphism ρ from Y to $\mathcal{E}_b(X)$. Thus we have

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[rr, bend right, "\delta"'] \arrow[rrr, bend left, dashed, "\nu\circ\delta=\tau"] & Y \arrow[r, "\rho"] & \mathcal{E}_b(X) \arrow[r, "\nu"] & Y \end{tikzcd}$$

Therefore, $\nu \circ \rho(\tau(x)) = \tau(x)$ for every $x \in X$, which, by Lemma 5.1.10 implies that $\nu \circ \rho = id_Y$. Similarly we can show that $\rho \circ \nu = id_{\mathcal{E}_b(X)}$. □

A consequence of the previous theorem is the following: in order that the H_b-H_b-envelope of X exist, it is necessary and sufficient that every function on $H_b(X)$ extends to a holomorphic function of bounded type on $\mathcal{E}_b(X)$. In the next section we will show that this is not always the case, so the H_b-H_b-envelope does not always exist.

5.2 Envelopes of open subsets of a Banach space

In this section we restrict ourselves to open subsets of a Banach space E. In order to give a more precise and concrete description of the H_b-envelope, we first study when every function in $H_b(U)$ can be extended to some larger open subset of E. We are particularly interested in establishing if the extensions are also of bounded type. As a consequence of the results in this section, we characterize the H_b-envelope of an open balanced set U in terms of the polynomially convex hulls of the U-bounded sets. We show that in general the extensions to the H_b-envelope are not of bounded type, answering a question of Hirschowitz [Hir72]. Since we have seen that the H_b-envelope is contained in the spectrum, extensions to the spectrum may also fail to be of bounded type. Also, the same example shows that the H_b-H_b-envelope of a balanced set does not always exist. However, we will see that if U is bounded and balanced then the extension is of bounded type and thus the H_b-envelope is also its H_b-H_b-envelope.

First we give some definitions: Let $U \subset E$ be an open set. Let $\mathcal{F}$ be a set of functions defined on U (e.g. $H_b(U)$, $H_b(E)$, or $\mathcal{P}(E)$), and A be a U-bounded set. We denote by $\widehat{A}_{\mathcal{F}}$ its $\mathcal{F}$-hull, that is

$$\widehat{A}_{\mathcal{F}} = \{x \in E : \ |f(x)| \leq \|f\|_A \ \text{ for every } f \in \mathcal{F}\}.$$

If $U_n = \{x \in U : \ \|x\| \leq n, \text{ and } dist(x, E \setminus U) \geq \frac{1}{n}\}$, then $\{U_n\}_n$ is a fundamental sequence of U-bounded sets, and we define the set

$$\widehat{U}_{\mathcal{F}} := \bigcup_{n\in\mathbb{N}} \left(\widehat{U}_n\right)_{\mathcal{F}}.$$

Definition 5.2.1. *U is $\mathcal{F}$-convex if $\widehat{A}_{\mathcal{F}}$ is U-bounded for every U-bounded set $A \subset U$.*

Our definition of $\mathcal{F}$-convex set coincides with the notion of strongly $\mathcal{F}$-convex set investigated by Vieira in [Vie07]. She proved that U is $\mathcal{P}(E)$-convex if and only if U is $H_b(E)$-convex (moreover, she proved that $\widehat{A}_{\mathcal{P}(E)} = \widehat{A}_{H_b(E)}$ for each bounded set A). If U is balanced then it is also equivalent for U to be $H_b(U)$-convex ([Vie07, Proposition 1.5]).

Inspired by [CGM05], we say that a point $x \in E$ is an evaluation for $H_b(U)$ if there is some $\varphi \in M_b(U)$ such that $f(x) = \varphi(f)$ for every $f \in H_b(E)$. If $H_b(E)$ is dense in $H_b(U)$ then φ is uniquely determined. In this case it will be denoted by δ_x. The set of all evaluation points for $H_b(U)$ will be denoted by $\overset{\vee}{U}$. So we have the following:

Proposition 5.2.2. *Let $\mathcal{F} = \mathcal{P}(E)$ or $H_b(E)$, then*

(1) *$U \subset \overset{\vee}{U} \subset \widehat{U}_{\mathcal{F}}$.*

(2) *If $\mathcal{P}(E)$ is dense in $H_b(U)$ (for example, if U is balanced), then $\overset{\vee}{U} = \widehat{U}_{\mathcal{F}}$.*

(3) *U is $\mathcal{F}$-convex if and only if $U = \overset{\vee}{U} = \widehat{U}_{\mathcal{F}}$.*

(4) *$\overset{\vee}{U}$ is an open subset of $\pi(\pi^{-1}(E))$.*

Proof. (1) If $z \notin \widehat{U}_{\mathcal{F}}$ then there exist $f_n \in \mathcal{F}$ such that $f_n(z) = 1$ and $\|f_n\|_{U_n} \leq \frac{1}{n}$.

In particular, $f_n \to 0$ in $H_b(U)$, and then $\psi(f_n) \to 0$ for every $\psi \in M_b(U)$. But if $\varphi \in M_b(U)$ is such that $\varphi(f) = f(z)$ for every $f \in H_b(E)$ then $\varphi(f_n) = f_n(z) = 1$ for every $n \in \mathbb{N}$, which is a contradiction. Therefore, $z \notin \overset{\vee}{U}$.

(2) If $\mathcal{P}(E)$ is dense in $H_b(U)$ and $z \in \widehat{U}_{\mathcal{F}}$, then there exists $n \in \mathbb{N}$ such that $|f(z)| \leq \|f\|_{U_n}$ for every $f \in H_b(E)$ and therefore δ_z is a bounded homomorphism defined on a dense subset of $H_b(U)$. Hence we can extend δ_z to an element of $M_b(U)$, and then $z \in \overset{\vee}{U}$.

(3) The "only if" part is a consequence of the definitions and from the first assertion. The "if" part follows from [Vie07, Lemma 1.3].

(4) Let $x \in \overset{\vee}{U}$, then there exist $\varphi \in M_b(U)$ such that $\varphi(f) = f(x)$ for every entire function of bounded type. Since $\overset{\vee}{U} \subset E$, φ actually belong to $\pi^{-1}(E)$.

Thus there exists $\delta > 0$ such that $\varphi^y \in \pi^{-1}(E)$ for every $y \in B_E(0, \delta)$ (see Lemma 5.1.4). Moreover for every $f \in H_b(E)$, $\varphi^y(f) = f(x + y)$. Therefore, $x + y$ is in $\overset{\vee}{U}$.

□

Corollary 5.2.3. *Let U be an open balanced subset of a Banach space E. Then $\widehat{U}_{\mathcal{P}}$ is the H_b-envelope of U.*

Proof. By Corollary 5.1.9 we have to show that $\delta(\widehat{U}_{\mathcal{P}})$ is the connected component of $\pi^{-1}(E) \subset M_b(U)$ which contains $\delta(U)$. The proof of Proposition 5.2.2 (2) actually shows that $\delta\big(\widehat{U}_{\mathcal{P}}\big) \subset M_b(U)$. Thus it is contained in $\pi^{-1}(E)$. Moreover, by [Vie07, Lemma 1.4], $\widehat{U}_{\mathcal{P}}$ is balanced and hence connected.

On the other hand, if $z \in E \setminus \widehat{U}_{\mathcal{P}}$, then for every $n \in \mathbb{N}$, there exist functions $f_n \in H_b(E)$ such that $\|f_n\|_{U_n} \leq \frac{1}{2^n}$ and $|f_n(z)| > 1$. Thus $f_n \to 0$ in $H_b(U)$ and therefore δ_z cannot be a continuous homomorphism. □

Thus, if U is balanced we can extend holomorphic functions of bounded type on U to $\widehat{U}_{\mathcal{P}}$. By Corollary 5.1.8 the inclusion $U \hookrightarrow \widehat{U}_{\mathcal{P}}$ is a strong H_b-extension. Moreover, it is possible to obtain extensions which are of bounded type "on every point" of $\widehat{U}_{\mathcal{P}}$ in the following sense.

Proposition 5.2.4. *Let U be a balanced open set. Then every holomorphic function of bounded type on U extend to $\widehat{U}_{\mathcal{P}}$. Moreover for each $y \in \widehat{U}_{\mathcal{P}}$, there exist a connected open set U_y, such that $\{y\} \cup U \subset U_y \subset \widehat{U}_{\mathcal{P}}$ and such that every holomorphic function of bounded type on U extend to a bounded type function on U_y.*

For the proof we will use the following two Lemma's which are similar to results in [Din71b].

Whenever polynomials are dense in $H_b(U)$, we can extend holomorphic functions of bounded type on U to $\widehat{U}_{\mathcal{P}}$ since by Proposition 5.2.2 (2), $\widehat{U}_{\mathcal{P}}$ may be embedded in $M_b(U)$. Let $f \in H_b(U)$ and $(P_n)_n$ a sequence of polynomials which converges to f, if we denote by $\tilde{f}$ its extension to $\widehat{U}_{\mathcal{P}}$, then it satisfies that $\tilde{f}(y) = \lim_n P_n(y)$ for every $y \in \widehat{U}_{\mathcal{P}}$.

Lemma 5.2.5. *Suppose that $\mathcal{P}(E)$ is dense in $H_b(U)$. Let B be a U-bounded set, $y \in \widehat{B}_{\mathcal{P}}$ and $f \in H_b(U)$. Then $\left\|\frac{d^k\tilde{f}(y)}{k!}\right\| \leq \sup\limits_{x\in B}\left\|\frac{d^k f(x)}{k!}\right\|$.*

Proof. Let $\phi \in \mathcal{P}^k(E)'$ such that $\|\phi\| \leq 1$. Then $g = \phi \circ \frac{d^k f}{k!}$ belongs to $H_b(U)$. Let $(P_n)_n \subset \mathcal{P}(E)$ such that $P_n \to g$ in $H_b(U)$. Then $\tilde{g}(y) = \lim P_n(y)$ and since $y \in \widehat{B}_{\mathcal{P}}$, we have that $|P_n(y)| \leq \|P_n\|_B$. Note also that $\tilde{g}$ and $\phi \circ \frac{d^k\tilde{f}}{k!}$ are holomorphic functions in $\widehat{U}_{\mathcal{P}}$ which coincide in U so they are the same function.

Therefore $\left|\phi\left(\frac{d^k\tilde{f}}{k!}(y)\right)\right| = |\tilde{g}(y)| \leq \|g\|_B = \sup_{x\in B}\left|\phi\left(\frac{d^k f(x)}{k!}\right)\right| \leq \sup_{x\in B}\left\|\frac{d^k f(x)}{k!}\right\|$. Since this is true for every $\phi \in \mathcal{P}^k(E)'$ such that $\|\phi\| \leq 1$, we conclude that $\left\|\frac{d^k\tilde{f}(y)}{k!}\right\| \leq \sup_{x\in B}\left\|\frac{d^k f(x)}{k!}\right\|$. □

Lemma 5.2.6. *Suppose that $\mathcal{P}(E)$ is dense in $H_b(U)$. Let $y \in \big(\widehat{U_n}\big)_{\mathcal{P}}$. Then for every function $f \in H_b(U)$ the Taylor series of $\tilde{f}$ at y converges on the ball $B_E(y, \frac{3}{4n})$. Moreover, $B_E(y, \frac{2}{3n}) \subset \widehat{U}_{\mathcal{P}}$ and for every $\|x\| < \frac{2}{3n}$, it holds that $|\tilde{f}(x+y)| \leq \|f\|_{U_{4n}}$.*

Proof. For every $x \in U_n$, we have that $B_E(x, \frac{3}{4n}) \subset U_{4n}$. By the previous lemma and Cauchy inequalities,

$$\left\|\frac{d^k\tilde{f}(y)}{k!}\right\| \leq \sup_{x\in U_n}\left\|\frac{d^k f(x)}{k!}\right\| \leq \|f\|_{U_{4n}}\left(\frac{4}{3}n\right)^k.$$

By the Cauchy-Hadamard formula, the Taylor series of $\tilde{f}$ at y converge in $B_E(y, \frac{3}{4n})$. If $\|x\| < \frac{2}{3n}$ then

$$\Big|\sum_{k=0}^{\infty}\frac{d^k\tilde{f}(y)}{k!}(x)\Big| \leq \sum_{k=0}^{\infty}\left\|\frac{d^k\tilde{f}(y)}{k!}\right\|\|x\|^k \leq \|f\|_{U_{4n}}\sum_{k=0}^{\infty}\left(\frac{4}{3}n\right)^k\left(\frac{2}{3n}\right)^k = 9\|f\|_{U_{4n}}.$$

Since this is true for every function in $H_b(U)$. In particular, for each $k \in \mathbb{N}$ and each polynomial P, we have that $\|P\|^k_{B_E(y,\frac{2}{3n})} = \|P^k\|_{B_E(y,\frac{2}{3n})} \leq 9\|P^k\|_{U_{4n}} = 9\|P\|^k_{U_{4n}}$ and thus $\|P\|_{B_E(y,\frac{2}{3n})} \leq \|P\|_{U_{4n}}$. This implies that $B_E(y, \frac{2}{3n}) \subset \big(\widehat{U_{4n}}\big)_{\mathcal{P}} \subset \widehat{U}_{\mathcal{P}}$. Therefore $\sum_{k=0}^{\infty}\frac{d^k\tilde{f}(y)}{k!}(x) = \tilde{f}(y+x)$ and a similar reasoning allows us conclude that $\|\tilde{f}\|_{B_E(y,\frac{2}{3n})} \leq \|f\|_{U_{4n}}$. □

Proof. (of Proposition 5.2.4) The point y belongs to $\big(\widehat{U_n}\big)_{\mathcal{P}}$ for $n \in \mathbb{N}$ sufficiently large, which is balanced by [Vie07, Lemma 1.4], and hence the segment $[0, y]$ is contained in $\big(\widehat{U_n}\big)_{\mathcal{P}}$. Let $U_y := \Big(\bigcup_{z\in[0,y]} B_E(z, \frac{2}{3n})\Big) \cup U$. By the last Lemma $U_y \subset \widehat{U}_{\mathcal{P}}$ and for each $f \in H_b(U)$, $\|\tilde{f}\|_{\bigcup_{z\in[0,y]} B_E(z,\frac{2}{3n})} \leq \|f\|_{U_{4n}} < \infty$. Therefore $\tilde{f}$ is of bounded type on U_y. □

Remark 5.2.7. Note that if $f \in H_b(U)$, then the extension $\tilde{f}$ to $\widehat{U}_{\mathcal{P}}$ belongs to the set $\{g \in H(\widehat{U}_{\mathcal{P}}) :\ g$ is bounded in $(\widehat{U_n})_{\mathcal{P}}$ for every $n \in \mathbb{N}\}$.

On the other hand, we cannot expect to extend the functions of $H_b(U)$ to connected subsets of E larger than $\widehat{U}_{\mathcal{P}}$ (or to points outside $\widehat{U}_{\mathcal{P}}$). Indeed suppose $V \supset U$ is another connected open set such that the inclusion $U \hookrightarrow V$ is an H_b-extension. If $z \in V$ then δ_z belongs to $M_b(U)$ by Corollary 5.1.8. Since for every entire function f, $\delta_z(f) = f(z)$, we conclude that z belongs to $\overset{\vee}{U}$. Therefore functions of bounded type cannot be extended outside $\overset{\vee}{U}$, and neither outside $\widehat{U}_{\mathcal{P}}$ by Proposition 5.2.2(1).

At this point, it is natural to ask if the extension to $\widehat{U}_{\mathcal{P}}$ must be of bounded type. By Corollary 5.2.3, for balanced subsets, this question coincides with the following question made by Hirschowitz in [Hir72, Remarque 1.8]: is the extension of every function of bounded type to the H_b-envelope of holomorphy of bounded type? The next example shows that in general the extensions to $\widehat{U}_{\mathcal{P}}$ are not necessarily in $H_b(\widehat{U}_{\mathcal{P}})$, answering both questions by the negative. Moreover, since by Theorem 5.1.9 the H_b-envelope is contained in the spectrum, this also shows that canonical extensions to the spectrum are not always of bounded type. We present, inspired in [CGM05, Example 7], an open balanced set $U \subset c_0$ and a function in $H_b(U)$ which cannot be extended to a holomorphic function of bounded type on $H_b(\widehat{U}_{\mathcal{P}})$.

Example 5.2.8. *There is an open balanced set $U \subset c_0$ and a function $g \in H_b(U)$ whose extension to $\widehat{U}_{\mathcal{P}(c_0)}$ is not of bounded type.*

Proof. Set $E = c_0$, and for $x \in c_0$, let

$$j(x) = \min\{j :\ |x_{2j}| = \max_{i\in\mathbb{N}} |x_{2i}|\}.$$

Note that $j(\lambda x) = j(x)$ if $\lambda \in \mathbb{C} \setminus \{0\}$. We define for $k > 4$

$$p_k(x) = \left|k x_{2k+1} + x_{2j(x)}\right| + k \sup_{i\neq k} |x_{2i+1}|,$$

and the sets $V_k = \{x \in c_0 :\ p_k(x) < 2\}$.

Let U be the following balanced set

$$U = \bigcup_{k>4} V_k + \frac{1}{4} B_{c_0},$$

where B_{c_0} denotes the open unit ball of c_0.

We first show that $\left\{\widehat{(U_n)}_{\mathcal{P}(c_0)}\right\}_n$ is not a fundamental sequence of $\widehat{U}_{\mathcal{P}(c_0)}$-bounded sets.

Fix $k > 0$. Since $p_k(e_{2k+1} - k e_{2m}) = 0$ then $e_{2k+1} - k e_{2m} \in V_k$ for every $m \in \mathbb{N}$. Thus if $\|x\| < \frac{1}{8}$, $dist(e_{2k+1} - k e_{2m} + x, c_0 \setminus U) > \frac{1}{8}$ and therefore $C_k := \{e_{2k+1} - k e_{2m} + \frac{1}{8} B_{c_0} :\ m \in \mathbb{N}\}$ is U-bounded.

We now prove that $e_{2k+1} + x \in \widehat{U}_{\mathcal{P}(c_0)}$ if $\|x\| < \frac{1}{8}$. Let $P \in \mathcal{P}(c_0)$ and $\varepsilon > 0$. Since finite type polynomials are dense in $\mathcal{P}(c_0)$, we can take $Q \in \mathcal{P}_f(c_0)$ such that $\|P - Q\|_{(k+2)B_{c_0}} < \varepsilon/3$. Moreover, since Q is of finite type and $\{e_{2m}\}_m$ is weakly null, there is m_0 such that if $m \geq m_0$ then $|Q(e_{2k+1} - k e_{2m} + x) - Q(e_{2k+1} + x)| < \varepsilon/3$. Thus

$$\begin{aligned} |P(e_{2k+1} - k e_{2m} + x) - P(e_{2k+1} + x)| \ \leq\ & |P(e_{2k+1} - k e_{2m} + x) - Q(e_{2k+1} - k e_{2m} + x)| \\ & + |Q(e_{2k+1} - k e_{2m} + x) - Q(e_{2k+1} + x)| \\ & + |Q(e_{2k+1} + x) - P(e_{2k+1} + x)| < \varepsilon. \end{aligned}$$

Therefore

$$|P(e_{2k+1}+x)| \leq \sup_{m\in\mathbb{N}} |P(e_{2k+1}-ke_{2m}+x)| \leq \sup_{y\in C_k} |P(y)|,$$

and so $e_{2k+1}+x \in \widehat{U}_{\mathcal{P}(c_0)}$ if $\|x\| < \frac{1}{8}$ and $k > 4$. This means that the set $D := \{e_{2n+1} : n > 4\}$ is $\widehat{U}_{\mathcal{P}(c_0)}$-bounded.

Suppose that $C \subset U$ is such that $D \subset \widehat{C}_{\mathcal{P}(c_0)}$. Now we show that C cannot be U-bounded. Since $e_{2n+1} \in \widehat{C}_{\mathcal{P}(c_0)}$, we have that $|P(e_{2n+1})| \leq \sup_{y\in C} |P(y)|$ for every $P \in \mathcal{P}(c_0)$. In particular, if $P = e'_{2n+1}$, this says that $1 \leq \sup_{y\in C} |y_{2n+1}|$. Thus there is a sequence $\{y^n\} \subset C$ such that $|y^n_{2n+1}| > \frac{3}{4}$.

For each y^n there exists $k > 4$ such that $y^n \in V_k + \frac{1}{4}B_{c_0}$ and thus there is some $x^n \in V_k$ such that $\|x^n - y^n\| < \frac{1}{4}$. Note that $|x^n_{2n+1}| > \frac{1}{2}$. Actually we have that $x^n \in V_n$, indeed, if $j \neq n$,

$$p_j(x^n) = \left|jx^n_{2j+1} + x^n_{2j(x^n)}\right| + j\sup_{i\neq j} |x^n_{2i+1}| \geq j|x^n_{2n+1}| > \frac{j}{2} > 2.$$

This implies that $x^n \notin V_j$. Then

$$2 > p_n(x^n) \geq \left|nx^n_{2n+1} + x^n_{2j(x^n)}\right| \geq \left|n|x^n_{2n+1}| - |x^n_{2j(x^n)}|\right|,$$

so we have that

$$\|x^n\| \geq |x^n_{2j(x^n)}| > n|x^n_{2n+1}| - 2 > \frac{n}{2} - 2. \tag{5.4}$$

Therefore $\|y^n\| > \|x^n\| - \frac{1}{4} > \frac{n}{2} - 2 - \frac{1}{4}$. Since $\{y^n\}_n \subset C$, this tell us that C is not bounded. We have proved that $\left\{\widehat{(U_n)}_{\mathcal{P}(c_0)}\right\}_n$ is not a fundamental sequence of $\widehat{U}_{\mathcal{P}(c_0)}$-bounded sets.

We now define the function g whose extension to $\widehat{U}_{\mathcal{P}(c_0)}$ is not of bounded type.

Let $g_n(x) := \left(\frac{5}{4}x_{2n+1}\right)^n$, for $x \in c_0$. We will prove that $\{g_n\}_n$ is a bounded sequence in $H_b(U)$ but not in $H_b(\widehat{U}_{\mathcal{P}(c_0)})$.

Since $g_n(e_{2n+1}) = \left(\frac{5}{4}\right)^n$, $\{g_n\}_n$ is not bounded in the $\widehat{U}_{\mathcal{P}(c_0)}$-bounded set D, and thus $\{g_n\}_n$ is not bounded in $H_b(\widehat{U}_{\mathcal{P}(c_0)})$.

Let A be U-bounded and take $M > 0$ such that $\|x\| < M - \frac{1}{4}$ for all $x \in A$. Let $x_0 \in A$, then $x_0 = y + z$ with $y \in V_k$ for some $k \geq 5$ and $\|z\| < \frac{1}{4}$. We claim that if $n > 2(M+2)$ then $|y_{2n+1}| \leq \frac{1}{2}$. Indeed, if $n \neq k$, then $k|y_{2n+1}| < 2$ and thus $|y_{2n+1}| < \frac{2}{5}$. On the other hand, if $n = k$ and $|y_{2n+1}| > \frac{1}{2}$, we can apply equation (5.4) to y, which implies that $\|y\| > \frac{n}{2} - 2$. But this contradicts the fact that $\|y\| < M$ and $n > 2(M+2)$. Therefore

$$|g_n(x)| = \left|\frac{5}{4}(y_{2n+1} + z_{2n+1})\right|^n \leq \left(\frac{5}{4}\left(\frac{1}{2} + \frac{1}{4}\right)\right)^n < 1,$$

for every $x \in A$ and every $n > 2(M+2)$. Since $\sup\{|g_n(x)| : x \in A, 1 \leq n \leq 2(M+2)\} < \infty$, we conclude that $\{g_n\}_n$ is a bounded sequence in $H_b(U)$.

If we take g the function

$$g(x) = \sum_{n\in\mathbb{N}} \left(\frac{8}{9}\right)^n \left(\frac{5}{4}x_{2n+1}\right)^n,$$

then g belongs to $H_b(U)$ but (its extension) does not belong to $H_b(\widehat{U}_{\mathcal{P}(c_0)})$. $\square$

The previous example also shows that the H_b-H_b-envelope of holomorphy does not exist in general:

Corollary 5.2.9. *The H_b-H_b-envelope of holomorphy does not always exist.*

Proof. Let U a balanced open set such that there exist $f \in H_b(U)$ whose extension to $\widehat{U}_{\mathcal{P}}$, $\tilde{f}$, is not of bounded type (take, for instance, the open subset of c_0 given in the previous example). If the H_b-H_b-envelope of U existed, by Theorem 5.1.11 it should coincide with $\mathcal{E}_b(U) = \widehat{U}_{\mathcal{P}}$. But this is impossible since the extension of f to $\widehat{U}_{\mathcal{P}}$ is not of bounded type. $\square$

Note that the set considered in Example 5.2.8 is unbounded. For bounded balanced domains, we see that everything works fine. To prove this we will use the following Lemma which states that the polynomial hull of a balanced set coincides with the intersection of its homogeneous polynomial hulls. This was noticed, for example, in [Sic85] for balanced sets in $\mathbb{C}^n$.

Lemma 5.2.10. *Let $V \subset E$ be a balanced set. Then $\widehat{V}_{\mathcal{P}} = \cap_{n\in\mathbb{N}} \widehat{V}_{\mathcal{P}_n}$, where $\widehat{V}_{\mathcal{P}_n} = \{x \in E : |P(x)| \leq \|P\|_V$ for every $P \in \mathcal{P}^n(E)\}$.*

Proof. We only need to prove that $\cap_{n\in\mathbb{N}} \widehat{V}_{\mathcal{P}_n} \subset \widehat{V}_{\mathcal{P}}$ since the other inclusion is clearly true for every set.

Let $z \in \cap_{n\in\mathbb{N}} \widehat{V}_{\mathcal{P}_n}$ and let $P \in \mathcal{P}(E)$ with $deg\ P = k$. For $n \in \mathbb{N}$ we have that $P^n = Q_0 + \cdots + Q_{nk}$, with $Q_j \in \mathcal{P}^j(E)$. By the Cauchy inequalities, $\|Q_j\|_V \leq \|P^n\|_V$, thus $|P^n(z)| = |\sum_{j=0}^{nk} Q_j(z)| \leq \sum_{j=0}^{nk} \|Q_j\|_V \leq \sum_{j=0}^{nk} \|P^n\|_V = (nk+1)\|P\|_V^n$. Therefore $|P(z)| \leq (nk+1)^{\frac{1}{n}} \|P\|_V$ for every $n \in \mathbb{N}$, which implies that $|P(z)| \leq \|P\|_V$. $\square$

Theorem 5.2.11. *Let $U \subset E$ be a bounded open balanced set, then every function in $H_b(U)$ can be extended to a holomorphic function of bounded type in $\widehat{U}_{\mathcal{P}}$.*

Proof. Let $f \in H_b(U)$. By Corollary 5.2.3, f can be extended to a holomorphic function $\tilde{f}$ on $\widehat{U}_{\mathcal{P}}$. We must show that $\tilde{f} \in H_b(\widehat{U}_{\mathcal{P}})$.

Since U is a bounded balanced set, $(\frac{n}{n+1}U)_{n\in\mathbb{N}}$ is a fundamental system of U-bounded sets. We will prove that $\left((\frac{n}{n+1}U)^{\wedge}_{\mathcal{P}}\right)_{n\in\mathbb{N}}$ is a fundamental system of $\widehat{U}_{\mathcal{P}}$-bounded sets. Let $A \subset \widehat{U}_{\mathcal{P}}$ be a $\widehat{U}_{\mathcal{P}}$-bounded set. Then $A \subset \frac{n}{n+1}\widehat{U}_{\mathcal{P}}$ for some $n \in \mathbb{N}$, so it suffices to prove that $\frac{n}{n+1}\widehat{U}_{\mathcal{P}} \subset (\frac{n}{n+1}U)^{\wedge}_{\mathcal{P}}$ for each $n \in \mathbb{N}$. Let $x \in \frac{n}{n+1}\widehat{U}_{\mathcal{P}}$, then $\frac{n+1}{n}x \in \widehat{U}_{\mathcal{P}}$. If $j \in \mathbb{N}$ and $Q_j \in \mathcal{P}^j(E)$ then $|Q_j(\frac{n+1}{n}x)| \leq \|Q_j\|_U$. Therefore $|Q_j(x)| \leq \left(\frac{n}{n+1}\right)^j \|Q_j\|_U = \sup_{y\in U} |Q_j(\frac{n}{n+1}y)| = \|Q_j\|_{\frac{n}{n+1}U}$, which means that $x \in (\frac{n}{n+1}U)^{\wedge}_{\mathcal{P}_j}$ for every j. By Lemma 5.2.10, $x \in (\frac{n}{n+1}U)^{\wedge}_{\mathcal{P}}$.

We have shown that, for each $\widehat{U}_{\mathcal{P}}$-bounded set A, it holds that $A \subset (\frac{n}{n+1}U)^{\wedge}_{\mathcal{P}}$ for some $n \in \mathbb{N}$. Thus, by Remark 5.2.7, $\tilde{f} \in H_b(\widehat{U}_{\mathcal{P}})$. $\square$

We already that $\widehat{U}_{\mathcal{P}}$ is the H_b-envelope of U in case U is a balanced open set. By the above theorem we have the following.

Corollary 5.2.12. *Let U be a bounded open balanced set of Banach space E. Then $\widehat{U}_{\mathcal{P}}$ is the H_b-H_b-envelope of U.*

We end this section by applying previous results to study H_b-domains of holomorphy:

Corollary 5.2.13. *Let $U \subset E$ be a bounded open balanced set, then $\widehat{U}_{\mathcal{P}}$ is an H_b-domain of holomorphy.*

Proof. Let $z \in \widehat{U}_{\mathcal{P}}$ and A be a U-bounded set such that $|\delta_z(f)| \leq \|f\|_A$ for every f in $H_b(U)$. Let $r < dist(A, U^c)$, then it can be shown as in [AGGM96] that $\{(\delta_z)^a \ : \ a \in E, \ \|a\| < r\} \subset M_b(U)$ (note that since we only consider $a \in E$ the symmetric regularity of E is not needed). But this means that $B_E(z, r) \subset \widehat{U}_{\mathcal{P}}$. Therefore $dist(z, (\widehat{U}_{\mathcal{P}})^c) \geq dist(A, U^c)$.

Now we can adapt the proof of [DV04, Proposition 2.4] (together with Theorem 5.2.11). $\square$

Next corollary is now an immediate consequence of the previous results.

Corollary 5.2.14. *Let $U \subset E$ be a bounded open balanced set. Then U is an H_b-domain of holomorphy if and only if $U = \widehat{U}_{\mathcal{P}}$.*

5.3 Extending functions of bounded type to open subsets of E''

In this section we try to extend functions of $H_b(U)$ to open sets in E'' containing U. First of all note that the argument given after Remark 5.2.7 is no longer true, since the restriction map fails to be injective. Therefore it is not clear which is (if there exists) the largest set on E'' to which we can obtain extensions of bounded type in the sense of Proposition 5.2.4.

Let us start by defining the following variation of the set $\overset{\vee}{U}$:

$$\overset{\vee}{U}{}'' := \{z \in E'' \ : \ \text{there is some } \varphi \in M_b(U) \text{ such that } \varphi(f) = \tilde{f}(z) \text{ for every } f \in H_b(E)\}.$$

Note that $\overset{\vee}{U} = \overset{\vee}{U}{}'' \cap E$ and $\overset{\vee}{U}{}'' \subset \pi(M_b(U))$ for every open set U. Analogously, we define for a U-bounded set A,

$$\widehat{A}''_{\mathcal{P}} = \{x'' \in E'' : \ |AB(f)(x'')| \leq \|f\|_A \ \text{ for every } f \in \mathcal{P}(E)\},$$

where $AB(f)$ denotes the Aron-Berner extension of f and let

$$\widehat{U}''_{\mathcal{P}} := \bigcup_{n\in\mathbb{N}} \left(\widehat{U}_n\right)''_{\mathcal{P}}.$$

We can prove as in Proposition 5.2.2 that if U is balanced then $\widehat{U}''_{\mathcal{P}} = \overset{\vee}{U}{}''$.

Before we go on, let us make clear that we cannot expect $\widehat{U}''_{\mathcal{P}}$ to be the largest open subset of E'' to which functions on $H_b(U)$ extend. For example, take a nonreflexive Banach space E that is complemented in its bidual E'', say $E'' = E \oplus M$. Denote by π_E the projection to E. Then every function $f \in H_b(U)$ can be extended to $\tilde{f} \in H_b(U \times M)$ by $\tilde{f} = f \circ \pi_E$. On the other hand, the Hahn-Banach theorem shows that $\widehat{U}''_{\mathcal{P}} \subset \overline{j_E(coe(U))}^{w^*}$. Thus, in general we can extend to sets which are larger than $\widehat{U}''_{\mathcal{P}}$. But we can see that if $j_E(U) \subset W \subset E''$ and we consider a continuous homomorphism $e : H_b(U) \to H(W)$ such that $e(f)(J_E(x)) = f(x)$ for every $x \in U$, $f \in H_b(U)$ and which coincides with the Aron-Berner extension on polynomials, then W must be a subset of $\widehat{U}''_{\mathcal{P}}$. Indeed, if $z \in W \setminus \widehat{U}''_{\mathcal{P}}$, then there exist functions $f_n \in H_b(E)$ such that $|AB(f_n)(z)| > 1$ and $\|f_n\|_{U_n} < \frac{1}{n}$. Then $f_n \to 0$ in $H_b(U)$ and thus $AB(f_n) = e(f_n) \to 0$ in $H(W)$, which contradicts the fact that $|AB(f_n)(z)| > 1$. This shows in particular that the Aron-Berner extension does not coincide with the composition with a projection π_E, at least for bounded sets. And this allows us to think of $\widehat{U}''_{\mathcal{P}}$ as a candidate to be the largest set in which the Aron-Berner extension is defined.

A continuous homomorphism $e : H_b(U) \to H(W)$ $(W \subset E'')$ which is an extension (i.e. $e(f)(J_E(x)) = f(x)$ for every $f \in H_b(U)$ and every $x \in U$) and which coincides with the Aron - Berner extension for polynomials will be called an *AB-extension homomorphism*. Note that for us, an AB-extension is a homomorphism between spaces of holomorphic functions, but in the framework of Riemann domains the extensions are morphisms with special properties. This motivates the following definitions:

Definition 5.3.1. Let (X, p) be a connected Riemann domain over E and (Y, q) a connected Riemann domain over the bidual E''. A continuous application $\tau : X \to Y$ is said to be an *AB-morphism* if $J_E(p(x)) = q(\tau(x))$ for every $x \in X$.

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ E \arrow[r, "J_E"] & E'' \end{tikzcd}$$

Definition 5.3.2. Let $\mathcal{F} \subset H(X)$ and $\mathcal{G} \subset H(Y)$. An AB-morphism τ is an *$\mathcal{F}$-$\mathcal{G}$-AB-extension* ($\mathcal{F}$-AB-extension if $\mathcal{G} = H(Y)$) if for each $f \in \mathcal{F}$ there exist a unique $\tilde{f} \in \mathcal{G}$ such that
i) $\tilde{f} \circ \tau = f$
ii) $\tilde{f}$ is locally the Aron-Berner extension of f; that is for each $x \in X$, $AB\big(f \circ (p_{|_{B(x,r)}})^{-1}\big) = \tilde{f} \circ (q_{|_{B(\tau(x),r)}})^{-1}$, for some $r > 0$.

Definition 5.3.3. Let $\mathcal{F} \subset H(X)$ be a topological algebra. An AB-morphism $\tau : X \to Y$ is a *strong $\mathcal{F}$-AB-extension* if it is an $\mathcal{F}$-AB-extension and the application $\mathcal{F} \ni f \mapsto \tilde{f}(y)$ is in the spectrum of $\mathcal{F}$.

Note that any H_b-H_b-AB-extension $\tau : X \to Y$ must be a strong H_b-AB-extension. Indeed if we let $\mathcal{F}$ be the set $\{\tilde{f} : f \in H_b(X)\}$, then it is a Fréchet space when considered with the topology of uniform convergence on Y-bounded sets. Thus the homomorphism $e : H_b(X) \to H_b(Y)$ determined by τ is a bijection from $H_b(X)$ to $\mathcal{F}$. Moreover, it is clear that e^{-1} is continuous. Therefore e must be continuous.

Remark 5.3.4. If U is an open set of E such that $\mathcal{P}(E)$ is dense in $H_b(U)$ and W is an open set of E'' such that $J_E(U) \subset W$, then $(J_E)_{|_U}$ is a strong H_b-AB-extension if and only if there exists a homomorphism $e : H_b(U) \to H(W)$ such that
i) $e(f)(J_E(x)) = f(x)$ for every $x \in U$, $f \in H_b(U)$,
ii) $H_b(U) \ni f \mapsto e(f)(z)$ belongs to $M_b(U)$ for every $z \in W$ and,
iii) e coincides with the Aron-Berner extension on polynomials, that is $AB(P) = e(P)$ for every polynomial $P \in \mathcal{P}(E)$.

Indeed, if $(J_E)_{|_U}$ is a strong H_b-AB-extension it is clear that $AB(P) = e(P)$ for every $P \in \mathcal{P}(E)$.
Conversely, let $(P_n)_n \subset H_b(U)$ such that $P_n \to f$ in $H_b(U)$ and $r > 0$ such that $B_E(x, r) \subset U$. Then for every $z \in B_{E''}(J_E x, r)$, $AB(P_n)(z) = \widetilde{P_n}(z) \to \tilde{f}(z)$ and thus $AB(f_{|_{B_E(x,r)}}) = \tilde{f}_{|_{B_{E''}(J_E x,r)}}$ for every $x \in U$, $f \in H_b(U)$.

Remark 5.3.5. In [DV04], AB-morphisms are called morphisms, and the authors also consider H_b-extensions, but their definition differ from ours. Indeed, they only ask for condition *i)* in the definition of AB-extension 5.3.2. Thus, for example, if $\pi_E : E'' \to E$ is a projection, then π_E is an H_b-extension in the context of [DV04], but it is not an AB-H_b-extension (see comments before Definition 5.3.1).

Recall [GGM93, Theorem 1.3] that if U is an absolutely convex open subset of then the Aron-Berner extension is an isometric isomorphism $AB : H^\infty(U) \to H^\infty(int(\overline{U}^{w^*}))$.

Corollary 5.3.6. *Let $U \subset E$ be an open absolutely convex bounded set. Then $\widehat{U}''_{\mathcal{P}} = int(\overline{U}^{w^*})$.*

Proof. $\widehat{U}''_{\mathcal{P}} \subset int(\overline{U}^{w^*})$ by the Hahn - Banach theorem. $int(\overline{U}^{w^*}) \subset \widehat{U}''_{\mathcal{P}}$ since by [GGM93, Theorem 1.5] there is an AB- extension morphism from U to $int(\overline{U}^{w^*})$. $\square$

By the comments before Definition 5.3.1, if W is an open set of E'' such that $J_E(U) \subset W$, then $(J_E)_{|U}$ is a strong H_b-AB-extension then W must be contained in $\widehat{U}''_{\mathcal{P}}$. We will prove bellow that it is the "AB-envelope":

Definition 5.3.7. The AB-morphism $\tau : X \to Y$ is said to be a *$\mathcal{F}$-AB-envelope of holomorphy* of X if τ is a $\mathcal{F}$-AB-extension of X and if for each $\mathcal{F}$-AB-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\mu \circ \nu = \tau$.

The AB-morphism $\tau : X \to Y$ is said to be a *strong $\mathcal{F}$-AB-envelope of holomorphy* of X if τ is a strong $\mathcal{F}$-AB-extension of X and if for each strong $\mathcal{F}$-AB-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\mu \circ \nu = \tau$.

The AB-morphism $\tau : X \to Y$ is said to be a *H_b-H_b-AB-envelope of holomorphy* of X if τ is a strong H_b-H_b-AB-extension of X and if for each H_b-H_b-AB-extension of X, $\nu : X \to Z$, there is a morphism $\mu : Z \to Y$ such that $\mu \circ \nu = \tau$.

$$\begin{tikzcd}
X \arrow[rr, "\nu"] \arrow[rrrr, bend left, "\tau"] \arrow[dr, "p"'] & & Z \arrow[rr, dashed, "\mu"] \arrow[dr] & & Y \arrow[dl, "q"] \\
& E \arrow[rr, "J_E"] & & E'' &
\end{tikzcd}$$

Let $U \subset E$ be an open subset. We want to fix an open subset of E'' to which there is an Aron-Berner extension. Although the following results should be known, we prefer to include them for self-containment, to fix notation and because we will use this construction in the next section. For $x \in U$ we denote $r_x = dist(x, U^c)$. Let $W \subset E''$ be the following open set

$$W = \bigcup_{x \in U} B_{E''}(x, \min\{\frac{r_x}{2}, 1\}).$$

Note that W is balanced if U is balanced and that $W \cap E = J_E(U)$.

Proposition 5.3.8. *Let $U \subset E$ be an open set such that $\mathcal{P}(E)$ is dense in $H_b(U)$. Then there is an AB- extension homomorphism from $H_b(U)$ to $H_b(W)$, that is $J_E|_U : U \to W$ is an H_b-H_b AB-extension.*

Note that by the comments above, $W \subset \widehat{U}''_{\mathcal{P}}$. Before we prove the proposition we need the following

Lemma 5.3.9. *Let C be a W-bounded set. Then there exists D, U-bounded, such that*

$$C \subset \bigcup_{x \in D} B_{E''}(x, \min\{\frac{r_x}{2}, 1\}).$$

Proof. Let $\varepsilon = dist(C, W^c)$ and $R = \sup\{\|z\| : z \in C\}$. Define $D := \{x \in U : \|x\| \leq R+1, r_x \geq \frac{2}{3}\varepsilon\}$ then D is U-bounded. We must show that $C \subset \bigcup_{x \in D} B_{E''}(x, \min\{\frac{r_x}{2}, 1\})$.

Let $z \in C$, and $x \in U$ such that $z \in B_{E''}(x, \min\{\frac{r_x}{2}, 1\})$, it suffices to prove that $x \in D$.

If $r_x < \frac{2}{3}\varepsilon$ then

$$dist(z, W^c) \leq dist(z, W^c \cap E) \leq \|z - x\| + r_x < \frac{r_x}{2} + r_x < \varepsilon,$$

which is a contradiction. Thus $r_x \geq \frac{2}{3}\varepsilon$.

Moreover, since $z \in B_{E''}(x,1)$ and $\|z\| \leq R$, we have that $\|x\| \leq R+1$. Therefore $x \in D$. $\square$

Proof. (of Proposition 5.3.8) Define $\tilde{\Phi} : \mathcal{P}(E) \to H_b(W)$ by $\tilde{\Phi}(P) = AB(P)_{|W}$. Let us show that it is continuous when we consider on $\mathcal{P}(E)$ the topology induced by $H_b(U)$. Take C a W-bounded set and $P \in \mathcal{P}(E)$. By the previous Lemma there is a U-bounded set D such that $C \subset \bigcup_{x\in D} B_{E''}(x, \min\{\frac{r_x}{2}, 1\})$. Clearly, the set $A = \bigcup_{x\in D} B_E(x, \min\{\frac{r_x}{2}, 1\})$ is U-bounded and since the Aron - Berner extension is isometric in $B_E(x, \min\{\frac{r_x}{2}, 1\})$ for every x, we have that

$$\|\tilde{\Phi}(P)\|_C \leq \|\tilde{\Phi}(P)\|_{\bigcup_{x\in D} B_{E''}(x,\min\{\frac{r_x}{2},1\})} = \|P\|_A < \infty.$$

Thus $\tilde{\Phi}$ extends to a continuous homomorphism $\Phi : H_b(U) \to H_b(W)$ which is clearly an AB-extension. $\square$

Proposition 5.3.10. *Let U be an open balanced set of a symmetrically regular Banach space E. Then*
(a) $J_E|_U : U \to \widehat{U}''_{\mathcal{P}}$ is a strong H_b-AB-extension. Every function in $H_b(U)$ can be extended to a holomorphic function in $\widehat{U}''_{\mathcal{P}}$ and the extensions are bounded on the sets $(\widehat{U}_n)''_{\mathcal{P}}$, for every $n \in \mathbb{N}$,
(b) if U is also bounded, there is an AB-extension homomorphism from $H_b(U)$ to $H_b(\widehat{U}''_{\mathcal{P}})$, that is $J_E|_U : U \to \widehat{U}''_{\mathcal{P}}$ is an H_b-H_b-AB-extension.

Proof. (a) We prove that $\widehat{U}''_{\mathcal{P}}$ is the connected component of $M_b(U)$ which contains U. Indeed, if $z \in \widehat{U}''_{\mathcal{P}}$ then δ_z is a bounded homomorphism when restricted to polynomials. Since polynomials are dense in $H_b(U)$ it follows that δ_z is in $M_b(U)$. Moreover we can easily modify the proof of [Vie07, Lemma 1.4] to show that $\widehat{U}''_{\mathcal{P}}$ is balanced. On the other hand if $z \in E'' \setminus \widehat{U}''_{\mathcal{P}}$ then for each $n \in \mathbb{N}$, there is a function $f_n \in H_b(U)$ such that $|AB(f_n)(z)| > \|f_n\|_{U_n}$ and thus $\delta_z \notin M_b(U)$.

It was proved in [DV04, Proposition 2.3] that bounded type functions extend to holomorphic functions on the spectrum $M_b(U)$ via the Gelfand transform, and this extension clearly coincides with the Aron-Berner extension for polynomials. Therefore the inclusion $J_E|_U$ is a strong H_b-AB-extension. The second assertion follows by the definition of the sets $(\widehat{U}_n)''_{\mathcal{P}}$ together with the density of polynomials on $H_b(U)$.

(b) Just adapt the proofs of Lemma 5.2.10 and Theorem 5.2.11. $\square$

Proposition 5.3.11. *Let U be an open (bounded) balanced set of a symmetrically regular Banach space E. Then $J_E|_U : U \to \widehat{U}''_{\mathcal{P}}$ is the strong H_b-AB-envelope (H_b-H_b-AB-envelope) of U.*

Proof. Let (Z,q) be a connected Riemann domain over E'' and $\tau : U \to Z$ a strong H_b-AB-extension. We have proved in the above proposition that $J_E|_U : U \to \widehat{U}''_{\mathcal{P}}$ is a strong H_b-AB-extension. Suppose that $q(Z) \subset \widehat{U}''_{\mathcal{P}}$, then clearly $q : Z \to \widehat{U}''_{\mathcal{P}}$ is a morphism and $J_E|_U = q \circ \tau$. Thus it suffices to prove that $q(Z) \subset \widehat{U}''_{\mathcal{P}}$.

Take $P \in \mathcal{P}(E)$ and let $\tilde{P}$ its AB-extension to Z. Then, for each $z \in Z$, $\tilde{P}(z) = AB(P)(q(z))$. Indeed if we define $Q(z) = AB(P)(q(z))$ then Q is holomorphic in Z. Moreover, for $x \in U$ and for sufficiently small $r > 0$, $Q \circ (q_{B(\tau(x),r)})^{-1}$ is the Aron-Berner extension of $P|_{B(x,r)}$, thus Q must be the AB-extension of P to Z.

Suppose now that $q(z) \notin \widehat{U}''_{\mathcal{P}}$ for some $z \in Z$. Then there exist $(P_n) \subset \mathcal{P}(E)$ such that $\|P_n\|_{U_n} \leq \frac{1}{n}$ and $|AB(P_n)(q(z))| > 1$. Thus $P_n \to 0$ in $H_b(U)$ but $|\tilde{P}_n(z)| = |AB(P_n)(q(z))| > 1$, so the evaluation at z cannot be a continuous homomorphism, which contradicts the fact that $\tau : U \to Z$ is a strong H_b-AB-extension.

If U is also bounded then by the previous proposition $J_E|_U : U \to \widehat{U}''_{\mathcal{P}}$ is an H_b-H_b-AB-extension and thus the H_b-H_b-AB-envelope of U. $\square$

Similarly to Corollary 5.2.13 we can prove the following.

Corollary 5.3.12. *Let E be a symmetrically regular Banach space. Let $U \subset E$ be a bounded open balanced set, then $\widehat{U}''_{\mathcal{P}}$ is an H_b-domain of holomorphy.*

5.4 Density of finite type polynomials

In several complex variables, the holomorphic convexity of U, or U being a domain of holomorphy, is equivalent to $M_b(U) = \delta(U)$. In our infinite dimensional setting this is not the case unless E has very particular properties. We can imprecisely explain this in the following way: if E is not reflexive, there are always elements of the bidual in the spectrum, so the equality $M_b(U) = \delta(U)$ cannot hold. On the other hand, if there are polynomials on E that are not weakly continuous on bounded sets, there is much more than evaluations in the spectrum [ACG91, AGGM96], and so $M_b(U) = \delta(U)$ is impossible even if E is reflexive. We will formalize this below, refining some results of [Vie07, Muj01].

In [Vie07], Vieira proved that for reflexive spaces such that every polynomial is approximable (i.e., for Tsirelson-like spaces), if U is a balanced $H_b(U)$-convex subset, then $M_b(U) = \delta(U)$. We now show that a converse of this theorem is an easy consequence of previous results.

Proposition 5.4.1. *Let E be a Tsirelson-like space and U a balanced open subset of E. Then U is $H_b(U)$-convex if and only if $M_b(U) = \delta(U)$.*

Proof. The "only if" part is Theorem 2.1 in [Vie07]. If $M_b(U) = \delta(U)$, then

$$\delta(U) \subset \delta(\overset{\vee}{U}) \subset \delta(\widehat{U}_{\mathcal{P}(E)}) \subset M_b(U) = \delta(U).$$

Therefore $\widehat{U}_{\mathcal{P}(E)} = U$ and thus U is $H_b(U)$-convex. $\square$

As many of our results, Proposition 5.4.1 holds for any U such that polynomials are dense in $H_b(U)$.

The following result was given in [Muj01, Theorems 1.1 and 1.2] for convex sets, but actually their proof works for balanced $H_b(U)$-convex sets:

Proposition 5.4.2. *Let U be any balanced $H_b(U)$-convex open subset of a Banach space E such that E' has the approximation property. Then E is a Tsirelson-like space if and only if $M_b(U) = \delta(U)$.*

Let us see that, in fact, the statement $M_b(U) = \delta(U)$ is equivalent to U being $H_b(U)$-convex subset of a Tsirelson-like space, thus obtaining an improvement of the previous result:

Theorem 5.4.3. *Let U be a balanced open subset of a Banach space E whose dual has the approximation property. Then $M_b(U) = \delta(U)$ if and only if E is a Tsirelson-like space and U is $H_b(U)$-convex.*

Proof. One implication follows from Proposition 5.4.1. For the converse, by the previous theorem, it suffices to prove that U is $H_b(U)$-convex. Since U is balanced this is equivalent to prove that U is $\mathcal{P}$-convex ([Vie07, Proposition 1.5]). By Proposition 5.2.2 (3) we must show that $U = \widehat{U}_{\mathcal{P}}$. Suppose that $w \in \widehat{U}_{\mathcal{P}} \setminus U$. Since by Corollary 5.1.8 the morphism $U \hookrightarrow \widehat{U}_{\mathcal{P}}$ is a strong H_b-extension it follows that δ_w belongs to $M_b(U)$. Therefore we cannot have the equality $M_b(U) = \delta(U)$. $\square$

As the previous theorem states, the equality $M_b(U) = \delta(U)$ is hard to achieve for domains in a Banach space E. This is because in general $M_b(U)$ cannot be identified with a subset of E. But we know that $M_b(U)$ can be projected on E'' via π, so a natural question is the following: suppose that U is $H_b(U)$-convex and E reflexive. Is it true that $\pi(M_b(U)) = U$? And if we drop off the reflexivity assumption, can we obtain something like $\pi(M_b(U)) = \overset{\vee}{U}''$ instead?

Let us see that, if finite polynomials are not dense, there are $\mathcal{P}$-convex subsets U for which $\pi(M_b(U))$ is larger than $\overset{\vee}{U}''$. In particular, if E is reflexive with the approximation property but not Tsirelson-like, there are subsets $U \subsetneq E$ that are $\mathcal{P}$-convex but $\pi(M_b(U)) = E$.

Proposition 5.4.4. *Let E be such that E' has the approximation property. The following conditions are equivalent:*
(i) finite type polynomials are dense in $H_b(E)$,
(ii) $\overset{\vee}{U}'' = \pi(M_b(U))$ for every open subset U of E,
(iii) $\overset{\vee}{U}'' = \pi(M_b(U))$ for every open $\mathcal{P}$-convex subset U of E.

If the conditions do not hold, there exists a proper subset U of E which is $H_b(U)$-convex but $E \subset \pi(M_b(U))$.

Proof. Suppose first that finite type polynomials are dense in $H_b(E)$. If $z \in \pi(M_b(U))$ then there is some $\varphi \in M_b(U)$ such that $\varphi(\gamma) = \gamma(z)$ for every $\gamma \in E'$. Since finite type polynomials are dense in $H_b(E)$ and φ is multiplicative, we have that $\varphi(f) = AB(f)(z)$ for every $f \in H_b(E)$, where $AB(f)$ denotes the Aron-Berner extension of f. Thus $z \in \overset{\vee}{U}''$. We have proved that (i) implies (ii).

Clearly, (ii) implies (iii).

$(iii) \Rightarrow (i)$: we will prove that if there is a n-homogeneous polynomial P which is not weakly continuous on bounded sets then we can find an open $\mathcal{P}$-convex set $U \subset E\backslash\{0\}$ such that $\overset{\vee}{U}'' \cap E = U$ but $E \subset \pi(M_b(U))$.

Take a weakly null bounded net $\{x_i\}_{i\in I} \subset S_E$ such that $P(x_i) > 1$ for every $i \in I$. Define the set

$$U = \Big\{x \in E : \; Re(P(x)) > \frac{1}{2}\Big\}.$$

For $y \in E \setminus U$, let $f_y(x) = \frac{1}{1-e^{P(y)-P(x)}}$. Then f_y is holomorphic in U. Moreover, let A be a U-bounded set and $R > 0$ such that $A \subset U_R$. Fix $x \in A \subset U_R \subset U$, let $t = Re(P(x))$ and take $\alpha > 0$ such that $Re(P(\alpha x)) = \alpha^n Re(P(x)) = \frac{1}{2}$ (which simply means that $\alpha = \big(\frac{1}{2t}\big)^{\frac{1}{n}} < 1$). Since αx does not belong to U, we have $\|x - \alpha x\| \geq \frac{1}{R}$ and substituting, we get $1 - \big(\frac{1}{2t}\big)^{\frac{1}{n}} \geq \frac{1}{R\|x\|} \geq \frac{1}{R^2}$. Therefore, $Re(P(x)) = t \geq \frac{1}{2}\big(\frac{R^2}{R^2-1}\big)^n$. Since $\big(\frac{R^2}{R^2-1}\big)^n > 1$, this implies that f_y is bounded on A, that is, $f_y \in H_b(U)$.

Thus, if we define for $k \in \mathbb{N}$, $f_k(x) = \sum_{m=0}^{k} e^{m(P(y)-P(x))}$, then $\{f_k\}$ is a bounded sequence in $H_b(U)$ since it converges to f. Moreover, $f_k \in H_b(E)$ for every $k \in \mathbb{N}$ and $f_k(y) = k+1$, which means that $y \notin \overset{\vee}{U}$. Therefore $\overset{\vee}{U}'' \cap E = U$.

Let $x \in E$, then we can find $\lambda > 0$ such that the set $\{x + \lambda x_i\}$ is U-bounded. Indeed, since $P(x+\lambda x_i) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k \overset{\vee}{P}(x^{n-k}, x_i^k)$, for $\lambda > 0$ big enough we have that $Re(\lambda^n P(x_i)) = \lambda^n P(x_i) \geq 1 + \big|\sum_{k=0}^{n-1} \binom{n}{k} \lambda^k \overset{\vee}{P}(x^{n-k}, x_i^k)\big|$ for every $i \in I$. Then $Re(P(x + \lambda x_i)) \geq 1$ for every $i \in I$. Take now $M > 0$ such that $\|x + \lambda x_i\| \leq M$ for every i. Take $y \in E \setminus U$. Then, if $\|y\| > M + 1$, it holds that

$\|x+\lambda x_i-y\|\geq 1$. On the other hand, if $\|y\|\leq M+1$, we have that

$$\begin{aligned}\frac{1}{2} &\leq |Re(P(x+\lambda x_i))-Re(P(y))|\leq |P(x+\lambda x_i)-P(y)|\\ &\leq \sum_{k=0}^{n-1}|\overset{\vee}{P}\big((x+\lambda x_i)^{n-k},y^k\big)-\overset{\vee}{P}\big((x+\lambda x_i)^{n-k-1},y^{k+1}\big)|\\ &\leq \|x+\lambda x_i-y\|\|P\|\sum_{k=0}^{n-1}\|x+\lambda x_i\|^{n-k-1}\|y\|^k\\ &\leq \|x+\lambda x_i-y\|\|P\|n(M+1)^{n-1}.\end{aligned}$$

Therefore, $\|x+\lambda x_i-y\|\geq \min\{1,\big(2\|P\|n(M+1)^{n-1}\big)^{-1}\}$ for every $y\in E\setminus U$, which implies that $\{x+\lambda x_i\}$ is U-bounded.

Then $\{x+\lambda x_i\}\subset U_R$ for some $R>0$ and since $\{x_i\}$ is weakly null, this means that $x\in\overline{U_R}^{w^*}$ and, by [CGM05, Proposition 18], $x\in\pi(M_b(U))$. Therefore $E\subset\pi(M_b(U))$.

It remains to prove that U is $\mathcal{P}$-convex. For this, if A is U-bounded, we can find as before $\varepsilon>0$ such that $A\subset\{x\in E:\ Re(P(x))>\frac{1}{2}+\varepsilon\}$, and if $y\notin U$, then $Re(P(y))\leq\frac{1}{2}$. Therefore if we set $f(x)=e^{-P(x)}$, $f\in H_b(E)$ and $|f(y)|\geq e^{-\frac{1}{2}}>e^{-\frac{1}{2}-\varepsilon}\geq\|f\|_A$, which means that $y\notin\widehat{A}_{H_b(E)}=\widehat{A}_{\mathcal{P}}$. Thus $\widehat{A}_{\mathcal{P}}\subset U$ and U is $\mathcal{P}$-convex. □

Corollary 5.4.5. *Let E be a reflexive space with the approximation property. The following conditions are equivalent:*
(i) E is a Tsirelson-like space,
(ii) $\hat{U}_{\mathcal{P}}=\pi(M_b(U))$ for every open subset U of E,
(iii) $U=\pi(M_b(U))$ for every open $\mathcal{P}$-convex subset U of E.

If the conditions do not hold, there exists a proper subset U of E which is $\mathcal{P}$-convex but $\pi(M_b(U))=E$.

If finite type polynomials are dense in $H_b(U)$, then π is clearly injective. Therefore, we have the following description of the spectrum of $H_b(U)$:

Corollary 5.4.6. *a) If finite type polynomials are dense in $H_b(E)$ and U is balanced, then $M_b(U)=\delta(\overset{\vee}{U''})$.*

b) If U is a balanced open subset of a Tsirelson-like space, then $M_b(U)=\delta(\overset{\vee}{U})$.

Again, reciprocal statements in the spirit of Theorem 5.4.3 are also valid.

We end this section with a Banach-Stone type result. In [Vie07, Theorem 3.1] the following was proved: if E and F are reflexive Banach spaces, one of them Tsirelson-like, and $U\subset E$, $V\subset F$ are open balanced and $\mathcal{P}$-convex, then the following conditions are equivalent:
(1) There exists a bijective mapping $g:V\to U$ such that $g\in H_b(V,U)$ (g is holomorphic and the image under g of V-bounded sets is U-bounded) and $g^{-1}\in H_b(U,V)$.
(2) The algebras $H_b(U)$ and $H_b(V)$ are topologically isomorphic.

In that case it follows that E and F are isomorphic Banach spaces.

In [CGM05, Corollary 22] a similar result was proved for convex balanced open sets when every polynomial on E'' (or F'') is approximable. In that case it follows that E' and F' are isomorphic.

We will slightly improve those results with the following (see below for precise definitions):

Theorem 5.4.7. *Let E, F be Banach spaces, $V \subset F$, $U \subset E$ open balanced subsets and suppose that every polynomial on E'' is approximable.*

If $\phi : H_b(U) \to H_b(V)$ is a Fréchet algebra isomorphism then there exists a biholomorphic function $g : \widehat{V}''_{\mathcal{P}} \to \widehat{U}''_{\mathcal{P}}$, with $g \in \mathcal{H}^\infty\big((\widehat{V_n})''_{\mathcal{P}}, (\widehat{U_n})''_{\mathcal{P}}\big)$ and $g^{-1} \in \mathcal{H}^\infty\big((\widehat{U_n})''_{\mathcal{P}}, (\widehat{V_n})''_{\mathcal{P}}\big)$, both locally w^*-w^* continuous such that $\widetilde{\phi f} = \tilde{f} \circ g$ for every $f \in H_b(U)$.

Conversely, if g is such a function then the operator $\phi : H_b(U) \to H_b(V)$ given by $\phi f = \tilde{f} \circ g|_V$ is a Fréchet algebra isomorphism.

In that case E' is isomorphic to F'.

To prove this Theorem we will need some preliminary results.

Let $V \subset F$ be a balanced open subset. Then by Proposition 5.3.8, there exists an open set $W \subset F''$ such that every function in $f \in H_b(V)$ extend to a function $\tilde{f} \in H_b(W)$. Throughout the rest of this section W will denote this subset.

Lemma 5.4.8. *Let E, F be Banach spaces, $V \subset F$ an open balanced subset and $U \subset E$ open. Suppose that $\phi : H_b(U) \to H_b(V)$ is a continuous and multiplicative operator. Then*

a) the mapping $g : W \to E''$, defined by $g(y'') = \pi(\delta_{y''} \circ \phi)$ is holomorphic.

b) if F is symmetrically regular then the mapping $g : \widehat{V}''_{\mathcal{P}} \to E''$, defined by $g(y'') = \pi(\delta_{y''} \circ \phi)$ is holomorphic.

Proof. a) Denote by $\theta_\phi : M_b(V) \to M_b(U)$ the restriction of the transpose of ϕ. Then g is just the composition $W \xrightarrow{\delta} M_b(V) \xrightarrow{\theta_\phi} M_b(U) \xrightarrow{\pi} E''$ which is well defined by Proposition 5.3.8. If we take $y'' \in W$ and $x' \in E'$, then $g(y'')(x') = \delta_{y''}(\phi x') = \widetilde{\phi x'}(y'')$. Thus g is weak*-holomorphic on W and therefore holomorphic (see for example [Muj86, Exercise 8D]).

b) By Proposition 5.3.10 we can define g on $\widehat{V}''_{\mathcal{P}}$. The proof of a) works fine. □

For an open set $U \subset E$, consider a family of subsets $A_k \subset U$, $k \in \mathbb{N}$, such that $\bigcup_k A_k = U$. We define:

$$\mathcal{H}^\infty\big((A_k)_k\big) = \{f \in H(U) : \ \|f\|_{A_k} < \infty \text{ for every } k\},$$

which is a Fréchet algebra with the topology of uniform convergence on the A_k's. If $(A_k)_k$ form a fundamental system of U-bounded sets, then we have $\mathcal{H}^\infty\big((A_k)_k\big) = H_b(U)$. Note that, if U is balanced, by Propositions 5.3.8 and 5.3.10 every function $f \in H_b(U)$ can be extended to a function $\tilde{f} \in \mathcal{H}^\infty\big((A_k)_k\big)$, where $A_k = (\widehat{U_k})''_{\mathcal{P}} \cap W$ or $A_k = (\widehat{U_k})''_{\mathcal{P}}$ in case E is symmetrically regular.

Also, if $V \subset F$ and we have a family of subsets $B_j \subset V$ such that $\bigcup_j B_j = V$, we define the Fréchet algebra

$$\begin{aligned} \mathcal{H}^\infty\big((A_k)_k, (B_j)_j\big) &= \{f \in H(U, V) : \ \text{there exists a subsequence } (n_k)_k \text{ such} \\ &\qquad \text{that } f(A_k) \subset B_{n_k} \text{ for every } k\}. \end{aligned}$$

If $(A_k)_k$ and $(B_j)_j$ form a fundamental system of U-bounded sets and V-bounded sets respectively, then $\mathcal{H}^\infty\big((A_k)_k, (B_j)_j\big)$ is simply $H_b(U, V)$.

Proposition 5.4.9. *Let E, F be Banach spaces, $V \subset F$, $U \subset E$ open balanced subsets and suppose that every polynomial on E is approximable. Let $\phi : H_b(U) \to H_b(V)$ is a continuous operator. Then*

a) ϕ is multiplicative if and only if there exists a holomorphic function $g : W \to \widehat{U}''_{\mathcal{P}}$, with $g \in \mathcal{H}^\infty\big((B_j)_j, (A_k)_k\big)$, where $A_k = (\widehat{U_k})''_{\mathcal{P}}$ and $B_k = (\widehat{V_k})''_{\mathcal{P}} \cap W$, such that $\widetilde{\phi f} = \tilde{f} \circ g$ for every $f \in H_b(U)$.

b) if F is symmetrically regular, ϕ is multiplicative if and only if there exists a holomorphic function $g : \widehat{V}''_{\mathcal{P}} \to \widehat{U}''_{\mathcal{P}}$, $g \in \mathcal{H}^\infty\big((B_j)_j, (A_k)_k\big)$, where $A_k = (\widehat{U_k})''_{\mathcal{P}}$ and $B_k = (\widehat{V_k})''_{\mathcal{P}}$ such that $\widetilde{\phi f} = \tilde{f} \circ g$ for every $f \in H_b(U)$.

Proof. *a*) Suppose that ϕ is multiplicative. Let g be the mapping defined by Lemma 5.4.8 *a*). By Corollary 5.4.6, the spectrum $M_b(U)$ can be identified with $\widehat{U}''_{\mathcal{P}}$, thus g maps W inside $\widehat{U}''_{\mathcal{P}}$ and, for $f \in H_b(U)$ and $y'' \in W$, we have $\tilde{f}(g(y'')) = \widetilde{\phi f}(y'')$ by the definition of g.

It remains to prove that $g \in \mathcal{H}^\infty\big((B_j)_j, (A_k)_k\big)$. Suppose that for some $n_0 \in \mathbb{N}$, $g(B_{n_0})$ is not contained in any of the A_k's. Thus there exist a sequence $(x''_k)_k = (g(y''_k))_k \subset g((\widehat{V}_{n_0})''_{\mathcal{P}} \cap W)$ such that $x''_k \notin (\widehat{U_k})''_{\mathcal{P}}$. This means there exist polynomials $P_k \in \mathcal{P}(E)$ such that $\|P_k\|_{U_k} < \frac{1}{2^k}$ and $\tilde{P}_k(x''_k) > k + \sum_{j=1}^{k-1} |\tilde{P}_j(x''_k)|$. Then $\sum P_k$ converges in $H_b(U)$ to a function f. Thus $\phi f \in H_b(V)$, so $\widetilde{\phi f}$ belongs to $\mathcal{H}^\infty\big((B_k)_k\big)$ and $\|\widetilde{\phi f}\|_{(\widehat{V}_{n_0})''_{\mathcal{P}} \cap W} < \infty$. This is a contradiction since $\|\widetilde{\phi f}\|_{(\widehat{V}_{n_0})''_{\mathcal{P}} \cap W} \geq |\widetilde{\phi f}(y''_k)| = |\tilde{f}(g(y''_k))| = |\tilde{f}(x''_k)| > k - 1$ for every k.

The converse is immediate.

b) The same proof as in *a*) but using Lemma 5.4.8 *b*) works. □

We will say that a function is *locally w^*-continuous* at a point x' of a dual Banach space if there exists a (norm) neighborhood such that the function restricted to this neighborhood is w^*-continuous. A function is locally w^*-continuous on an open set if it is locally w^*-continuous at each point of the set.

Proof. (of Theorem 5.4.7) Suppose that ϕ is an isomorphism. Let $g \in \mathcal{H}^\infty\big((B_j)_j, (A_k)_k\big)$ be the application given by Proposition 5.4.9 *a*), and let $h : \widehat{U}''_{\mathcal{P}} \to F''$ be the holomorphic map obtained in Lemma 5.4.8 *b*) using the homomorphism ϕ^{-1} (note that our hypothesis imply that E is symmetrically regular). Then $h \circ g$ is the composition

$$W \xrightarrow{\delta} M_b(V) \xrightarrow{\theta_\phi} M_b(U) \xrightarrow{\pi} \widehat{U}''_{\mathcal{P}} \xrightarrow{\delta} M_b(U) \xrightarrow{\theta_{\phi^{-1}}} M_b(V) \xrightarrow{\pi} F''.$$

Since, by Corollary 5.4.6, $M_b(U) = \delta(\widehat{U}''_{\mathcal{P}})$, it follows that $h \circ g = id_W$. Thus $dh(g(0)) \circ dg(0) = id_{F''}$ and therefore F'' is isomorphic to a complemented subspace of E'' which implies that every polynomial on F'' is approximable and, in particular, that F is symmetrically regular. Thus we can use Proposition 5.4.9 *b*) and define g in $\widehat{V}''_{\mathcal{P}}$ and we have $h \circ g = id_{\widehat{V}''_{\mathcal{P}}}$. By Corollary 5.4.6 we have $M_b(V) = \delta(\widehat{V}''_{\mathcal{P}})$, thus $g \circ h = id_{\widehat{U}''_{\mathcal{P}}}$ which means that $h = g^{-1}$. By Proposition 5.4.9 *b*), g^{-1} belongs to $\mathcal{H}^\infty\big((\widehat{U_n})''_{\mathcal{P}}, (\widehat{V_n})''_{\mathcal{P}}\big)$.

Moreover, for every $x \in E$, $x \circ g = \widetilde{\phi x}$ is locally w^*-continuous since it is locally an Aron-Berner extension. Therefore g is locally w^*-w^* continuous on V. Similarly g^{-1} is locally $w^* - w^*$ continuous.

Conversely, suppose that g is as above. Define $\phi f = \tilde{f} \circ g|_V$ for $f \in H_b(U)$ and $\psi h = \tilde{h} \circ g^{-1}|_U$ for $h \in H_b(V)$. Then clearly $\phi : H_b(U) \to H_b(V)$ and $\psi : H_b(V) \to H_b(U)$ are continuous and multiplicative. We want to prove that $\psi = \phi^{-1}$. Let $f \in H_b(U)$, then

$$\psi \circ \phi f = \psi(\tilde{f} \circ g|_V) = \widetilde{\tilde{f} \circ g|_V} \circ g^{-1}|_U. \tag{5.5}$$

Note that $\tilde{f} \circ g$ belongs to $\mathcal{H}^\infty\big((B_j)_j\big)$ and is locally w^*-continuous (since every polynomial on E is approximable, the Aron-Berner extension of f is w^*-continuous). Thus for each $z \in \widehat{V}''_{\mathcal{P}}$, applying

[ACG95, Lemma 2.1] to $\tilde{f} \circ g$ restricted to a suitable ball implies that $\frac{d^k(\tilde{f}\circ g)(z)}{k!}$ is w^*-continuous, and therefore by [Zal90, Theorem 2] we can conclude that $\frac{d^k(\tilde{f}\circ g)(z)}{k!}$ is in the image of the Aron-Berner extension and thus $\widetilde{\tilde{f} \circ g|_V} = \tilde{f} \circ g$. Then we obtain from (5.5) that $\psi \circ \phi f = f$. Similarly, $\phi \circ \psi h = h$ for $h \in H_b(V)$.

It remains to prove that E' and F' are isomorphic.

Differentiating $g \circ g^{-1}$ at 0 we obtain that E'' and F'' are isomorphic. Applying [ACG95, Lemma 2.1] to $y'' \mapsto g(y'')(x')$ restricted to a suitable ball, we obtain that the differential of g at any point is w^*-w^*-continuous (and analogously for g^{-1}). Therefore, the isomorphism between E'' and F'' is the transpose of an isomorphism between F' and E'. □

If F is complemented in its bidual (for example, if F is a dual space) the previous theorem holds if every polynomial on E (and not necessarily on E'') is approximable. Indeed we obtain as in the proof of the theorem that F'' is isomorphic to a complemented subspace of E'', and there we can easily prove that F is isomorphic to a complemented subspace of E. We can then conclude the theorem with the same proof.

We also have:

Corollary 5.4.10. *Let E, F be Banach spaces, one of them Tsirelson-like; $V \subset F$, $U \subset E$ open balanced and bounded subsets. Then $\phi : H_b(U) \to H_b(V)$ is a Fréchet algebra isomorphism if and only if there exists a biholomorphic function $g \in H_b(\widehat{V}_{\mathcal{P}}, \widehat{U}_{\mathcal{P}})$ such that $g^{-1} \in H_b(\widehat{U}_{\mathcal{P}}, \widehat{V}_{\mathcal{P}})$ and the operator $\widetilde{\phi f} = \tilde{f} \circ g$ for every $f \in H_b(U)$.*

In that case E and F are isomorphic Banach spaces.

The Tsirelson-James space T_J^* is not reflexive (it is not a Tsirelson-like space) but satisfies the conditions of Theorem 5.4.7 by [DGMZ04, Lemma 19].

5.5 On the Spectrum of $H_b(U)$

A consequence of Example 5.2.8 is that the canonical extension of a function in $H_b(U)$ is not necessarily of bounded type on the spectrum. The following proposition gives an equivalent condition for these extensions to be of bounded type (this should be compared to [DV04, Proposition 2.5]):

Proposition 5.5.1. *Let U be an open set of a simmetrically regular Banach space E. Then every function $f \in H_b(U)$ extends to a function $\tilde{f}$ of bounded type on $M_b(U)$ if and only if given any $M_b(U)$-bounded set B there exists a U-bounded set D such that $\varphi \prec D$ for all $\varphi \in B$.*

Proof. We have a well defined extension morphism

$$\begin{array}{rccc} e: & H_b(U) & \to & H_b(M_b(U)) \\ & f & \mapsto & \tilde{f}. \end{array}$$

Suppose that $f_n \to 0$ in $H_b(U)$ and that $\tilde{f}_n \to g$ in $H_b(M_b(U))$. Let $\varphi \in M_b(U)$, then we have that $g(\varphi) = \lim \tilde{f}_n(\varphi) = \lim \varphi(f_n) = 0$. By the Closed Graph Theorem, the map e is continuous. This means that given a $M_b(U)$-bounded set B there exists a U-bounded set D and a constant $c > 0$ such that $\|\tilde{f}\|_B \le c\|f\|_D$ for every function $f \in H_b(U)$. In particular, if $\varphi \in B$ then $|\varphi(f)| \le c\|f\|_D$ for every $f \in H_b(U)$. Since φ is multiplicative, we conclude that $\varphi \prec D$.

For the converse, let B be a $M_b(U)$-bounded set and D such that if $\varphi \in B$ then $\varphi \prec D$. Then $|\tilde{f}(\varphi)| = |\varphi(f)| \le \|f\|_D$ and therefore $\|\tilde{f}\|_B \le \|f\|_D < \infty$. □

In [AGGM96] the following inequality was implicitly shown:

$$\sup\left\{dist(A, U^c) : \varphi \prec A\right\} \leq dist_{M_b(U)}(\varphi). \tag{5.6}$$

If for some U we have equality or at least a reverse inequality with some constant, then extensions to $M_b(U)$ would be of bounded type, as a consequence of the previous proposition. We do not know of many examples in which extensions to $M_b(U)$ are of bounded type. One such example is a balanced and bounded open subset U of a Tsirelson-like space. In this case, by Corollary 5.2.12, the extensions to $M_b(U)$ are of bounded type. When $U = E$ and finite type polynomials are dense in $\mathcal{P}(E)$, we have that $M_b(E) = E''$. Thus the extension to the spectrum is the Aron-Berner extension, which is of bounded type. Moreover, if E is any symmetrically regular Banach spaces, it was shown in [Din99, Proposition 6.30] that the extension to the spectrum is of bounded type on each sheet of the spectrum. We now show that this does not imply that the extension to the spectrum is of bounded type on the whole Riemann domain.

Proposition 5.5.2. *Let E be a symmetrically regular Banach space such that there exists a continuous polynomial on E which is not weakly continuous on bounded sets. Then there exists a homogeneous polynomial whose extension to the spectrum $M_b(E)$ is not of bounded type.*

Proof. We may suppose that P is n-homogeneous whose restriction to a ball is not weakly continuous at 0. Let $(x_\alpha)_{\alpha\in\Delta}$ be a weakly null bounded net and $\varepsilon > 0$ such that $|P(x_\alpha)| > \varepsilon$ for every $\alpha \in \Delta$. Take a filter base on Δ, $\mathcal{B} = \{\alpha \in \Delta : \alpha \geq \alpha_0\}_{\alpha_0\in\Delta}$ and let Γ be an ultrafilter such that $\mathcal{B} \subset \Gamma$. For each $k \in \mathbb{N}$, define $\varphi_k(f) = \lim_\Gamma f(kx_\alpha)$, for $f \in H_b(E)$. Then φ_k is in $M_b(E)$. Moreover $\varphi_k(x') = 0$ for every $x' \in E'$ and thus $\pi(\varphi_k) = 0$ for every k. This implies that the set $C = \{\varphi_k : k \in \mathbb{N}\}$ is $M_b(E)$-bounded. But, $|\tilde{P}(\varphi_k)| = |\varphi_k(P)| = |\lim_\Gamma P(kx_\alpha)| \geq k^n\varepsilon$ and therefore $\|\tilde{P}\|_C = \infty$, that is, $\tilde{P}$ is not of bounded type on $M_b(E)$. $\square$

For the general case, recall that the open set in Example 5.2.8 was neither bounded nor convex, so one might ask if for the unit ball of a symmetrically regular Banach space things are easier. We do not know if in this case extensions to the spectrum are of bounded type, but we can answer for the negative the question on the reverse inequality in (5.6): fixed $1 < p < \infty$, there cannot be a constant $c > 0$ such that $\sup\left\{dist(A, B_{\ell_p}^c) : \varphi \prec A\right\} \geq c\, dist_{M_b(B_{\ell_p})}(\varphi)$ for every $\varphi \in M_b(B_{\ell_p})$.

For the following proposition, recall [AGGM96] that, for $\varphi \in M_b(B_{\ell_p})$, with $1 < p \leq \infty$ (for $p = 1$ we lose the symmetric regularity) and if $0 < r < 1$ is such that $\varphi \prec rB_{\ell_p}$ then for each $z \in \ell_p$ with $\|z\| < \frac{1}{1-r}$, we can define

$$\varphi^z(f) = \sum_{n=0}^{\infty} \varphi\left(y \mapsto \frac{d^n f(y)}{n!}(z)\right).$$

It is shown in [AGGM96, Section 2] that $\varphi^z(f)$ belongs to $M_b(B_{\ell_p})$. Moreover, the different mappings of the form $z \mapsto \varphi^z$ give the local section of π that defines the analytic structure of $M_b(B_{\ell_p})$. In fact, inequality (5.6) is a consequence of this: since $\varphi^z(f)$ is defined whenever $\|z\| < \frac{1}{1-r}$, we have $dist_{M_b(B_{\ell_p})}(\varphi) \geq \frac{1}{1-r}$, which is precisely the distance from rB_{ℓ_p} to $\ell_p \setminus B_{\ell_p}$.

In the sequel, for $\varphi \in M_b(U)$ we define $S(\varphi)$, the sheet of φ, as the connected component of $M_b(U)$ that contains φ.

Proposition 5.5.3. *If $1 < p < \infty$, then*

$$\inf_{\varphi \in M_b(B_{\ell_p})} \frac{\sup\left\{dist(A, B_{\ell_p}^c) : \varphi \prec A\right\}}{dist_{M_b(B_{\ell_p})}(\varphi)} = 0.$$

In other words, there is no reverse inequality in (5.6).

Proof. Set $U = B_{\ell_p}$. Let Γ be an ultrafilter on $\mathbb{N}$ containing all the sets of the form $\{n, n+1, n+2, \dots\}$ and define $\varphi_t \in M_b(U)$ by $\varphi_t(f) = \lim_\Gamma f(\frac{e_n}{t})$, with $t > 1$.

Take $z \in \ell_p$ with $\|z\| < \big(1 - (\frac{1}{t})^p\big)^{\frac{1}{p}}$. Then there is some $r > 1$ and $n_0 \in \mathbb{N}$ such that the set $A = \{sz + \frac{e_n}{t} : n \in \mathbb{N},\, n \geq n_0,\, |s| = r\}$ is U-bounded. By the Cauchy inequality,

$$\left|\frac{d^k f}{k!}(\frac{e_n}{t})(z)\right| \leq \frac{1}{r^k} \sup_{|s|=r} |f(\frac{e_n}{t} + sz)| \leq \frac{1}{r^k}\|f\|_A.$$

Therefore,

$$|\varphi_t^z(f)| = \left|\sum_{k=0}^{\infty} \varphi(\frac{d^k f(\cdot)}{k!}(z))\right| \leq \frac{1}{1-r}\|f\|_A$$

We have shown that $\varphi_t^z \in M_b(U)$. Hence $\pi(S(\varphi_t)) \supset \big(1 - (\frac{1}{t})^p\big)^{\frac{1}{p}} B_{\ell_p}$ and $dist_{M_b(U)}(\varphi_t) \geq \big(1 - (\frac{1}{t})^p\big)^{\frac{1}{p}}$.

If $r < \frac{1}{t}$ and $A \subset rB_{\ell_p}$, take a natural number $m > p$ and define $g_N(x) = \Big(t^m \sum_k x_k^m\Big)^N$. Then $\varphi_t(g_N) = 1$ for every $N \in \mathbb{N}$, but $\|g_N\|_A \leq \|g_N\|_{rB_{\ell_p}} \leq (tr)^N \to 0$ as $N \to \infty$. Thus $\varphi_t \not\prec A$. Finally, since clearly $\varphi_t \prec \frac{1}{t}B_{\ell_p}$ we have that $\sup\{dist(A, U^c) : \varphi_t \prec A\} = 1 - \frac{1}{t}$.

Therefore $\displaystyle\lim_{t\to 1^+} \frac{\sup\Big\{dist(A, B_{\ell_p}^c) : \varphi_t \prec A\Big\}}{dist_{M_b(B_{\ell_p})}(\varphi_t)} = 0$ and the result follows. $\square$

In the proof of the previous proposition we have shown that $\pi(S(\varphi_t)) \supset \big(1 - (\frac{1}{t})^p\big)^{\frac{1}{p}} B_{\ell_p}$. If $p \in \mathbb{N}$, Proposition 5.5.4 below show that we have moreover an equality: $\pi(S(\varphi_t)) = \big(1 - (\frac{1}{t})^p\big)^{\frac{1}{p}} B_{\ell_p}$.

This means that for any $\varphi \in M_b(B_{\ell_p})$ defined as in the Proposition, the sheet of φ is an analytic copy (via π) of a ball centered at 0. It can be seen that for a convex and balanced open subset U of a symmetrically regular Banach space E, $\pi(M_b(U))$ coincides with $int(\overline{U}^{w^*})$ (see, for example, [CGM05, Lemma 20]). The previous example shows that for U the unit ball of ℓ_p, some sheets are projected into proper subsets of $int(\overline{U}^{w^*}) = U$. Therefore, $M_b(B_{\ell_p})$ cannot be seen as a union of disjoint copies of B_{ℓ_p}, as one might have thought from the case $U = E$, where $M_b(E)$ is a disjoint union of analytic copies of E''.

We now show that if we restrict ourselves to a distinguished part of the spectrum of B_{ℓ_p}, with p a natural number greater than 1, then the extension of the functions in $H_b(B_{\ell_p})$ is of bounded type.

Take any block basis $(x_n)_{n\in\mathbb{N}}$ in the unit ball of ℓ_p with $\|x_n\| \to r \in (0, 1)$ and consider, as usual, an ultrafilter Γ on $\mathbb{N}$ containing all the sets of the form $\{n, n+1, n+2, \dots\}$. Let $\varphi \in M_b(B_{\ell_p})$ given by,

$$\varphi(f) = \lim_\Gamma f(x_n), \tag{5.7}$$

for $f \in H_b(B_{\ell_p})$. Note that, since block bases are weakly null, we have $\varphi \in \pi^{-1}(0)$.

Let us define the following subdomain of $M_b(B_{\ell_p})$:

$$M_b^0(B_{\ell_p}) = \bigcup_{\varphi \text{ given by } (5.7)} S(\varphi).$$

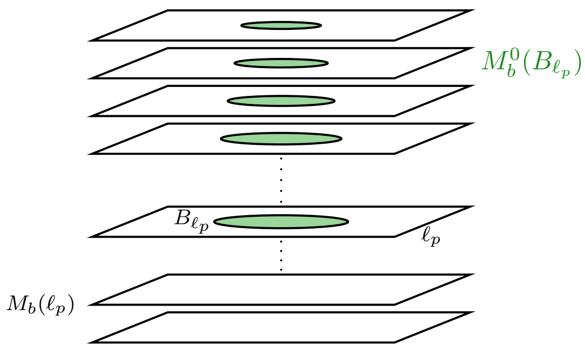
Note that all adherent points of the sequence $(\delta_{te_n})_n$ $(0 < t < 1)$ belong to $M_b^0(B_{\ell_p})$, so the number of connected components of $M_b^0(B_{\ell_p})$ has at least the cardinality of $\beta\mathbb{N}$. Moreover, it is not clear that there are morphisms in $M_b(B_{\ell_p})$ that are not in $M_b^0(B_{\ell_p})$ (though to assert such a thing one should be able to prove of a really strong Corona theorem for $H_b(B_{\ell_p})$). One might argue that morphisms in $M_b(B_{\ell_p})$ can be built with sequences that are not blocks or with nets, but it is not clear that those cannot have an alternative representation as in (5.7). Anyway, $M_b^0(B_{\ell_p})$ is a relatively large part of $M_b(B_{\ell_p})$, where "relatively" should be understood as "up to our knowledge".

Note also that if we consider bounded type entire functions on ℓ_p, then a slight modification of Proposition 5.5.2 may be used to prove that there exists a homogeneous polynomial whose extension to the distinguished spectrum of $H_b(\ell_p)$ is not of bounded type.

Let us first describe the sheet of a homomorphism in $M_b^0(B_{\ell_p})$.

Proposition 5.5.4. *Let p be a natural number greater than 1 and let $\varphi \in M_b(B_{\ell_p})$ be given as in (5.7). Then $\pi(S(\varphi)) = (1 - r^p)^{\frac{1}{p}} B_{\ell_p}$.*

Thus, the distinguished spectrum of B_{ℓ_p} may be thought of as a union of balls of decreasing radius:



Proof. (of Proposition 5.5.4) Take $z \in \ell_p$ with $\|z\| < \left(1 - r^p\right)^{\frac{1}{p}}$. Since $(x_n)_n$ is a block sequence with respect to $(e_k)_k$, then, for some n_0, $A = \{z + x_n : n \in \mathbb{N}, n \geq n_0\}$ is U-bounded. Since $\varphi^z(f) = \lim_\Gamma f(x_n + z)$, we have $\varphi^z \prec A$ and thus $\varphi^z \in M_b(B_{\ell_p})$. Hence $\pi(S(\varphi)) \supset \left(1 - (r)^p\right)^{\frac{1}{p}} B_{\ell_p}$.

For the reverse inclusion let $\|z\| > \left(1 - r^p\right)^{\frac{1}{p}}$. Let $(\alpha_k) \subset \mathbb{C}$ and $(j_n) \subset \mathbb{N}$ an increasing sequence such that $x_n = \sum_{k=j_n+1}^{j_{n+1}} \alpha_k e_k$.

Then for some $\delta > 0$, $\|z\|^p + r^p > 1 + \delta$ and since $\|x_n\| \to r$, there exists $M \in \mathbb{N}$ such that for every $n > M$, $\|z\| + \|x_n\| > 1 + \delta$ and such that

$$\Big| \sum_{k=1}^{M} |z_k|^p + \sum_{k=j_n+1}^{j_{n+1}} \big|\alpha_k + z_k\big|^p + \sum_{\substack{k>M \\ k \notin \{j_n+1,\dots,j_{n+1}\}}} z_k^p \Big| > 1 + \delta.$$

Let us define $f_N(x) = \left(\sum_{k=1}^{\infty} \theta_k x_k^p\right)^N$, where $|\theta_k| = 1$ and $\theta_k z_k^p = |z_k|^p$ if $1 \leq k \leq M$, $\theta_k(z_k + \alpha_k)^p = |z_k + \alpha_k|^p$ for every $j_n + 1 \leq k \leq j_{n+1}$ and $n > M$, and $\theta_k = 1$ otherwise. Then since $\|f_N\|_U \leq 1$ for every N, $\{f_N : N \in \mathbb{N}\}$ is a bounded sequence in $H_b(B_{\ell_p})$ and for every $n > M$,

$$|f_N(z + x_n)| = \Big| \sum_{k=1}^{M} |z_k|^p + \sum_{k=j_n+1}^{j_{n+1}} \big|\alpha_k + z_k\big|^p + \sum_{\substack{k>M \\ k \notin \{j_n+1,\dots,j_{n+1}\}}} z_k^p \Big|^N > (1 + \delta)^N.$$

Since f_N is a polynomial, $\varphi^z(f_N) = \lim_\Gamma f_N(z + x_n)$. Therefore $|\varphi^z(f_N)| = \lim_\Gamma |f_N(z + x_n)| \geq (1 + \delta)^N$, which implies that $\varphi^z \notin M_b(B_{\ell_p})$. Thus $(1 - r^p)^{\frac{1}{p}} B_{\ell_p} \subset \pi(S(\varphi)) \subset (1 - r^p)^{\frac{1}{p}} \overline{B_{\ell_p}}$, but since it must be open, we conclude that $\pi(S(\varphi)) = (1 - r^p)^{\frac{1}{p}} B_{\ell_p}$. □

The distinguished spectrum $M_b^0(B_{\ell_p})$ is an open subset of $M_b(B_{\ell_p})$, since it is the union of some connected components of $M_b(B_{\ell_p})$. Thus $M_b^0(B_{\ell_p})$ is a Riemann domain over ℓ_p and every function $f \in H_b(B_{\ell_p})$ extends to a holomorphic function $\tilde{f}$ on $M_b^0(B_{\ell_p})$. We now show that this extension is of bounded type.

Proposition 5.5.5. *If p be a natural number greater than 1, for any $f \in H_b(B_{\ell_p})$, its extension $\tilde{f}$ to $M_b^0(B_{\ell_p})$ is of bounded type.*

Proof. Let $\varepsilon > 0$ and take the $M_b^0(B_{\ell_p})$-bounded set $A = \{\phi \in M_b^0(B_{\ell_p}) : dist_{M_b^0(B_{\ell_p})} \geq \varepsilon\}$. By Proposition 5.5.4, A only intersects the sheets such that $\pi(S(\varphi)) = (1-r^p)^{\frac{1}{p}} B_{\ell_p}$ with $(1-r^p)^{\frac{1}{p}} \geq \varepsilon$. Let $\varphi \in A$ such that $\pi(\varphi) = 0$, then $A \cap S(\varphi) = \{\varphi^z : \|z\| \leq (1 - r^p)^{\frac{1}{p}} - \varepsilon\}$. Thus there exists $\delta > 0$ such that $\|z\|^p + r^p < 1 - \delta$ for every $\varphi^z \in A$.

Let $\varphi \in A$, $\pi(\varphi) = 0$, and let $(x_n)_n$ and Γ be a block sequence and an ultrafilter defining φ (that is, $\varphi(f) = \lim_\Gamma f(x_n)$). Since $\|x_n\| \to r$, $\|z\|^p + \|x_n\|^p < 1 - \delta$ if n is big enough. Moreover, there is $n_0 \in \mathbb{N}$ such that $\|z + x_n\|^p < 1 - \delta$ for every $n \geq n_0$ since $(x_n)_n$ is a block sequence. Then $\{x_n + z : n \geq n_0\}$ is contained in the U-bounded set $(1 - \delta)^{\frac{1}{p}} B_{\ell_p}$ and $\varphi^z \prec (1 - \delta)^{\frac{1}{p}} B_{\ell_p}$.

Therefore $\|\tilde{f}\|_A = \sup_{\phi \in A} |\phi(f)| \leq \|f\|_{(1-\delta)^{\frac{1}{p}} B_{\ell_p}} < \infty$. □

Note that Proposition 5.5.3 can be restated with the infimum taken for $\varphi \in M_b^0(B_{\ell_p})$, since the homomorphisms that were used in the proof were defined by constant multiples of the canonical basis. As a consequence, there is no reverse inequality in (5.6) even if we restrict ourselves to $M_b^0(B_{\ell_p})$. Therefore, the absence of such a reverse inequality should not be thought as an impediment for the extensions to the whole spectrum to be of bounded type. This means that, if there are extensions to $M_b(B_{\ell_p})$ that fail to be of bounded type, the proof of this fact will probably not be based on the absence of this reverse inequality.

Since by Proposition 5.5.4 the connected components of $M_b^0(B_{\ell_p})$ are balls, we have the following corollary (which could also be deduced from the previous proposition):

Corollary 5.5.6. *If p be a natural number greater than 1, then $M_b^0(B_{\ell_p})$ is an H_b-domain of holomorphy.*

We would like to finish with the following questions and comments:

- Are the extension of bounded type functions on B_{ℓ_2} to the spectrum $M_b(B_{\ell_2})$ of bounded type?
- Even more, we don't know any example of an open set of a Banach space which satisfy that the extension of bounded type functions to the spectrum is of bounded type, unless finite type polynomials are dense in all the polynomials.
- Neither can we answer the following question: given a bounded open set U (non-balanced) are the extensions to the H_b-envelope of bounded type?

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Index